Outer contact billiards

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Abstract

We introduce outer contact billiards as an odd dimensional counterpart to outer symplectic billiards. Until now, outer billiards have only been considered in even dimensional symplectic vector spaces. By projectivizing outer symplectic billiards we obtain outer contact billiards, where the affine midpoint condition descends to its projective analog, namely harmonic conjugation.

We prove that outer contact billiards generates contactomorphisms. For quadratic surfaces in $\mathbb{RP}^3$, we show that the correspondence is completely integrable: its domain is foliated by invariant quadrics and, on each leaf, the dynamics is determined by the iteration of an explicit linear transformation. We also establish two rigidity results for periodic trajectories: outer contact billiards admit no 3-periodic orbits and, among quadratic tables, only one admits 4-periodic orbits.

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1 Introduction

In 1959, B. Neumann ([17]) introduced the outer billiard map, a dynamical system defined outside a convex planar domain. Unlike the classical Birkhoff billiard, where the trajectory evolves inside the table by reflecting at its boundary, the outer billiard map acts on the exterior of the table: a point is sent to its reflection across a tangency point of the table (see Figure 1). Motivated by questions concerning the stability of the solar system, J. Moser continued the study of the subject in the 1970s ([15], [16]), initiating a long line of research that continues to this day.

In [22], S. Tabachnikov introduced outer symplectic billiards, extending the construction from the plane to arbitrary linear symplectic spaces. In this setting, the characteristic direction determined by the symplectic structure replaces the tangent line of the planar construction, yielding a natural higher-dimensional analog of outer billiards. This topic has been further studied in [24], [1], and [2], among others.

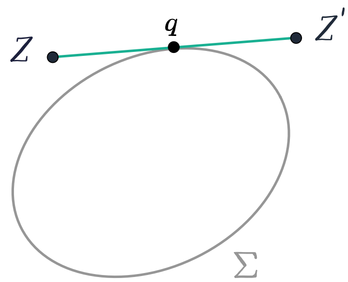

Figure 1: The outer billiard map in the plane.

Since symplectic manifolds are necessarily even-dimensional, the present paper addresses the following natural question: *what should outer billiards look like in odd dimensions?*

Following the Arnold-Givental philosophy: that contact geometry is to symplectic geometry as projective geometry is to affine geometry (see [3], [21]), we propose an answer to this question. This analogy suggests that the outer symplectic billiards should admit a projective counterpart. Starting with a symplectic vector space (V, ω), consider its projectivization $\mathbb{P}V$. Its projectivization carries a canonical contact structure: the contact hyperplane at a given line through the origin is its symplectic complement. From this perspective, the passage from outer symplectic billiards to outer contact billiards amounts to replacing the affine ingredient of the construction, the midpoint condition, with its projective counterpart, namely harmonic conjugation. In other words, outer contact billiards are a scaling reduction of outer symplectic billiards (see, for example, [6]). This leads to a correspondence in projective space, which we call the *outer contact billiard correspondence*: given a hypersurface Σ, a pair of points, $Z, Z' \in \mathbb{P}V$, are in correspondence over Σ if, first of all, they lie on some characteristic line of Σ. For instance, in $\mathbb{RP}^3$ the characteristic lines of a surface are the lines of intersection between the tangent and contact planes as the points run over the surface.

Furthermore, each characteristic line of Σ contains two distinguished points: q, q' (see Definition 2.2) and we ask the points Z, Z', to be harmonic conjugates with respect to these distinguished points q, q' (see Definition 2.3, or Figure 8 below).

Here we will describe this correspondence explicitly when $\Sigma \subset \mathbb{RP}^3$ is a quadratic surface. Also we show in general (see §5 below for more precise statements) that:

Theorem. *The outer contact billiard correspondence generates local contactomorphisms. Moreover, it is the only choice of outer billiard generating local contactomorphisms over any table.*

Besides its simple geometric nature, this construction also has a natural dynamical motivation. A hypersurface in a contact manifold inherits a certain characteristic line field, i.e., first-order differential equation, whose integral curves play a central role in contact geometry and appear naturally, for example, in describing blow-ups of singularities in mechanical systems (see e.g. [20], [6]). The outer contact billiard correspondence provides discrete dynamics built along these same characteristic lines. In other words, contact billiards is a certain type of contact integrator (see e.g. [7], [8]) well adapted to the projective geometry of our situation. One expects contact billiard orbits to capture qualitative features of the corresponding integral curves, a behavior which is already apparent in the quadratic examples considered in Section 3.

The paper is organized as follows. In Section 2, we recall the projective geometric notions needed to define the correspondence. Section 3 contains some explicit examples associated with quadratic hypersurfaces, illustrating both the similarities and the differences between outer contact and outer symplectic billiards. We show that every quadratic example is completely integrable: the domain is filled by invariant leaves, and on each one, the billiard map is described by a fixed linear transformation. Appendix A shows that, up to a change of symplectic coordinates, our list of examples covers all quadratic tables in $\mathbb{RP}^3$. In Section 4, we study periodic orbits; in particular, we prove that the correspondence admits no three-periodic orbits (Proposition 4.1), and that there is only one quadratic table admitting a 4-periodic orbit in $\mathbb{RP}^3$ (Proposition 4.2). Section 5 establishes that the correspondence preserves the ambient contact structure (Theorem 5.1). Although projective geometry has no intrinsic notion of midpoint, two natural settings do: affine charts and a spherical chart. We therefore ask for which hypersurfaces the harmonic conjugate construction agrees with the corresponding midpoint construction used in outer symplectic billiards in these models. Theorems 5.2 and 5.3 provide complete characterizations in the affine and spherical settings, respectively. Finally, Section 6 collects several open problems. We hope that the present work serves as a first step toward a broader theory of outer contact billiards.

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2 Outer contact billiards

We first fix notation and recall some standard notions from projective (see e.g. [19]), and contact geometry (see e.g. [3]).

Let V be a vector space. Its projective space $\mathbb{P}V$ is the space of lines through the origin of V. We denote projective points by $Z \in \mathbb{P}V$ and write $\boldsymbol{Z} \in Z \setminus \{0\} \subset V$ for a choice of lift.

If $A, B, C, D \in \mathbb{P}V$ are collinear, one can always choose lifts $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in V$ so that $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. Such lifts are unique up to simultaneous rescaling. If $\boldsymbol{D} = \lambda \boldsymbol{A} + \eta \boldsymbol{B}$ then the quotient

$$\mathrm{cr}(A, B, C, D) := \frac{\lambda}{\eta},$$

is independent of the chosen lifts and its called the *cross ratio* of A, B, C, D. In some affine coordinate along the line containing the quadruple of collinear points it is given by the usual formula $\mathrm{cr}(A, B, C, D) = \frac{(A-C)(B-D)}{(A-D)(B-C)}$.

The points C and D are said to be *harmonic conjugates* with respect to A and B when $\mathrm{cr}(A, B, C, D) = -1$, or equivalently, if we can write $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$ (see Figure 2).

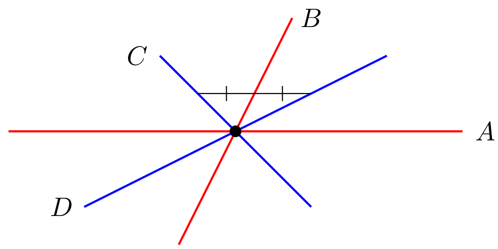


Figure 2: A harmonic tuple: C, D are harmonic conjugates with respect to A, B (and vice-versa).

Now, suppose the given vector space is also equipped with a symplectic form ω. The projective space inherits a contact structure ξ, assigning to a line through the origin of V its symplectic orthogonal complement (a hyperplane containing said line), that is:

Definition 2.1. *The* contact plane *through $q \in \mathbb{P}V$ is the plane*

$$\xi_q = q^\omega := \{\mathbf{u} \in V : \omega(\mathbf{u}, \mathbf{q}) = 0\} \in \mathbb{P}V^*.$$

Likewise, given a hyperplane $\Pi \in \mathbb{P}V^*$, its contact point *is the unique point $q_\Pi = \Pi^\omega \in \Pi$ satisfying*

$$\Pi = \xi_{q_\Pi}.$$

A Legendrian line *in $(\mathbb{P}V, \xi)$ is a line contained in some ξ_q and passing through q.*

Throughout the paper, we will often work with a specific situation:

Definition 2.2. *If $\Sigma \subset \mathbb{P}V$ is a regular hypersurface and $q \in \Sigma$, we write $q' = (T_q\Sigma)^\omega$ for the contact point of the tangent hyperplane $T_q\Sigma$. When $q' \neq q$, the* characteristic line *at $q \in \Sigma$ is the line joining q and q'.*

In computations, we may often use the following observation (immediate from the definitions)

Proposition 2.1. *For a hypersurface given (locally) as $\Sigma = f^{-1}(0) \subset \mathbb{P}V$ for $f : V \dashrightarrow \mathbb{R}$ some homogeneous function, then $q' = span\{X_f(q)\}$ for X_f the symplectic gradient of f: $\omega(\cdot, X_f) = df(\cdot)$.*

We are now ready to introduce the outer contact billiard correspondence (see Figure 3).

Definition 2.3. *Let $\Sigma \subset (\mathbb{P}V, \xi)$ be a hypersurface (the* table*). A pair of points $Z, Z' \in \mathbb{P}V$ are in* outer contact billiards correspondence *over Σ if there exist $q \in \Sigma$ such that*

1. *The line $Z \wedge Z'$ is the characteristic line at q, and*
2. *Z, Z' are harmonic conjugates with respect to q, q'.*

Since outer contact billiards are, in general, a correspondence rather than a map, we will call broadly:

Definition 2.4. *An* orbit *is a sequence $(Z_j)_{j\in\mathbb{Z}}$ such that each consecutive pair is in correspondence. An orbit is* periodic *if there exists a positive integer k such that $Z_{j+k} = Z_j$ for all j. The smallest such k is called the* period *of the orbit.*

For instance, we will not worry here about when there is some orientation convention which could direct orbits in some standard way, and, unless mentioned otherwise, will be interested in non-collinear orbits.

Remark 1. *A point $q \in \Sigma$ satisfying $T_q\Sigma = \xi_q$ is called a* singular point*. At such a point, the contact point of the tangent hyperplane coincides with q itself, that is, $q = q'$. Consequently, the characteristic line is no longer determined, and a further analysis (blowup) of the contact billiards relation around such points is necessary.*

Since contact structures are maximally non-integrable, one expects the singular set to be small; indeed, for a generic hypersurface, the singular points are isolated, and the set of singular points always has an empty interior. On the other hand, topological obstructions may force their existence. For instance, if Σ is a 2-sphere, then by the hairy ball theorem, the characteristic line field cannot be everywhere non-vanishing, and therefore Σ necessarily contains singular points.

Remark 2. *The contact structure $(\mathbb{P}V, \xi)$ is the model example of a contact reduction of $(V\backslash 0, \omega)$ with respect to the radial scaling symmetry [6]. Our definition of outer contact billiards is, in this sense, a scaling reduction of symplectic billiards along the cone spanned by Σ in V. See Figure 3.*

Remark 3. *Outer contact billiards depend only on the conformal class of the linear symplectic structure (V, ω).*

Remark 4. *We also propose (§6 below) an analogous definition for an* inner contact billiards *in $\mathbb{RP}^3$, whose study we will leave for future work.*

We often consider the linear symplectic structure on V as an identification

$$\omega : V \to V^*, \quad \boldsymbol{v} \mapsto \iota_{\boldsymbol{v}}\omega$$

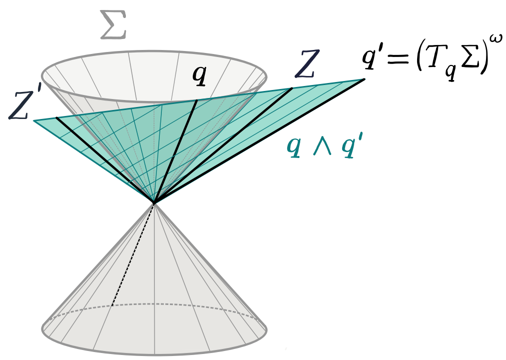


Figure 3: The outer contact billiard correspondence is obtained by projectivizing the outer symplectic billiard correspondence on the cone over the table. The contact point $q' = (T_q\Sigma)^\omega \in \mathbb{P}V$, and $q \in \Sigma$ generate the characteristic line $q \wedge q'$, along with the involution of $q \wedge q'$ by taking harmonic conjugates, $Z \longleftrightarrow Z'$, with respect to q, q' (as shown in Figure 2).

or projectively as a *polarity* on $\mathbb{P}V$, identifying (see Definition 2.1)

$$\mathbb{P}V \to \mathbb{P}V^*, \quad q \mapsto \xi_q.$$

Through this identification any hypersurface $\Sigma \subset \mathbb{P}V$ has a corresponding *polar surface*, $\Sigma' \subset \mathbb{P}V$, upon identifying its dual $\{\mathrm{Ann}(T_q\Sigma) : q \in \Sigma\} \subset \mathbb{P}V^*$ with a locus in $\mathbb{P}V$ through the above polarity. In summary:

Definition 2.5. *The* polar surface, $\Sigma' \subset \mathbb{P}V$, *with respect to* ω *of a hypersurface* $\Sigma \subset \mathbb{P}V$ *is the locus of contact points of* Σ*:*

$$\Sigma' := \{(T_q\Sigma)^\omega : q \in \Sigma\} \subset \mathbb{P}V.$$

The polar surface of a table $\Sigma \subset \mathbb{P}V$ plays a key role in its induced outer contact billiards.

3 Examples

Before presenting some general results, §5, we examine several examples in the first nontrivial case, when $\dim V = 4$. So, we will denote here $(\mathbb{R}^4, \omega)$ as some 4-dimensional linear symplectic vector space with contact reduction $(\mathbb{RP}^3, \xi)$ and will focus our attention on quadratic tables

$$\Sigma = \{Q = 0\} \subset \mathbb{RP}^3,$$

where $Q : \mathbb{R}^4 \to \mathbb{R}$ is some indefinite quadratic form. For such tables, their outer contact billiard admits an explicit description: the domain of the correspondence is filled by invariant quadrics, with an induced dynamics on each leaf determined by iterating a fixed linear transformation (see Propositions A.1, A.2 below). We first present those examples which we noticed to have some distinguished properties. The remaining types of quadratic surfaces and their contact dynamics are given explicitly in Section A.1 from Appendix A below.

Unless mentioned otherwise, we work in appropriate symplectic coordinates

$$\mathbb{R}^4 \ni (x,y,u,v), \qquad \omega = dx \wedge dy + du \wedge dv, \tag{1}$$

and, whenever convenient, identify

$$\mathbb{R}^4 \ni (x,y,u,v) \longleftrightarrow (x+iy, u+iv) = (z,w) \in \mathbb{C}^2. \tag{2}$$

3.1 Clifford tori

We consider first a table of the form

$$\Sigma_c = \{|z|^2 = c|w|^2\}$$

in the complex coordinates (2), for $c > 0$ some constant. This table projects not only under real scaling to a torus in $\mathbb{RP}^3$, but also under complex scaling (the Hopf map) to a circle in $\mathbb{CP}^1$.

First, we determine the polar surface of Σ_c. From Proposition 2.1, we compute the symplectic gradient: the contact point of $q = \mathrm{span}\{(z,w)\} \in \Sigma_c$ is $q' = \mathrm{span}\{(iz, -icw)\} \in \Sigma'$ so that

$$\Sigma' = \{c|z|^2 = |w|^2\}.$$

To visualize the correspondence, we pass to the double cover $S^3 \xrightarrow{2:1} \mathbb{RP}^3 = S^3/\{\pm 1\}$. The intersection $\Sigma_c \cap S^3$ is the Clifford torus

$$\boldsymbol{\Sigma} = \{(c_1 e^{i\theta}, c_2 e^{i\varphi}) \in \mathbb{C}^2 : \theta, \varphi \in S^1\} \subset S^3,$$

where the constants c_1, c_2 such that $c = (c_1/c_2)^2$ are normalized to satisfy $c_1^2 + c_2^2 = 1$. The projections of $\boldsymbol{\Sigma}$ on each complex coordinate are circles of radii c_1 and c_2, respectively. Likewise,

$$\boldsymbol{\Sigma}' = \{(c_2 e^{i\theta}, c_1 e^{i\varphi}) \in \mathbb{C}^2 : \theta, \varphi \in S^1\} \subset S^3.$$

Two points $\boldsymbol{Z}, \boldsymbol{Z}' \in S^3$ are in outer contact correspondence (see Definition 2.3) if there exist $\boldsymbol{q} \in \boldsymbol{\Sigma}$ with corresponding $\boldsymbol{q}' \in \boldsymbol{\Sigma}'$ and $t \in \mathbb{R}$ such that

$$\boldsymbol{Z} = \cos(t)\boldsymbol{q} + \sin(t)\boldsymbol{q}', \qquad \boldsymbol{Z}' = \cos(t)\boldsymbol{q} - \sin(t)\boldsymbol{q}'. \tag{3}$$

Projecting onto each complex coordinate, the characteristic geodesic becomes a tangent ellipse to the projections of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$, as shown in Figure 4.

Remark 5. *When $c_1 = c_2$, every characteristic line is contained in Σ_1, i.e., Σ_1 is 'self-dual'. Consequently, the domain of the correspondence reduces to Σ_1, a torus ruled by characteristic lines, and any pair of points lying on a common ruling are in outer contact billiard correspondence.*

For $c_1 \neq c_2$, the correspondence has the following properties:

1. The correspondence is defined on the region bounded by Σ_c and Σ'.
2. Two points $\boldsymbol{Z}, \boldsymbol{Z}' \in S^3$ are in outer contact correspondence if and only if they lie on the same characteristic geodesic and are equidistant from the tangency point. This mirrors the definition of outer symplectic billiards described in [22]. As we shall prove in Section 5.2, this equivalence characterizes Hopf tables.

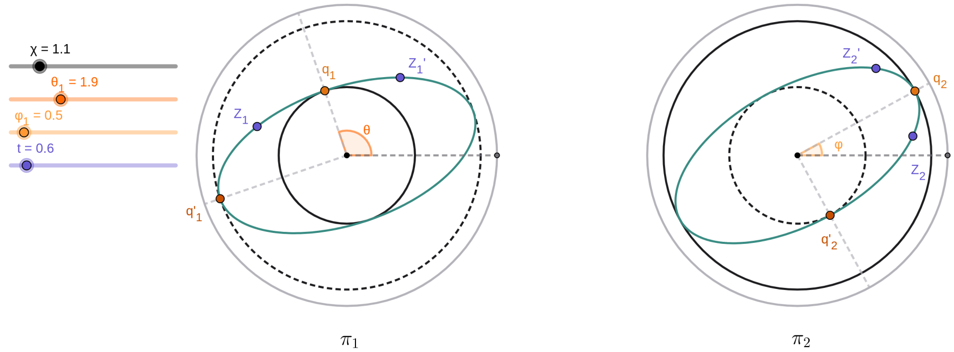


Figure 4: Outer contact billiard on a Clifford torus. The left and right images show the projections onto the first and second complex coordinates, respectively. The points $\boldsymbol{Z} = (Z_1, Z_2)$ and $\boldsymbol{Z'} = (Z'_1, Z'_2)$ are in outer contact billiard correspondence (note: the norms $|Z_1|, |Z_2|$ are not independent but rather related by $|Z_2|^2 = 1-|Z_1|^2$). An interactive version of this figure is available at GeoGebra.

3. Every point of the domain is in correspondence with exactly two points. The next point in the orbit is given by the rotations on each complex coordinate through angles α, β, where

$$\cos\alpha = \frac{c^2\cos^2 t - \sin^2 t}{c^2\cos^2 t + \sin^2 t}, \qquad \cos\beta = \frac{\cos^2 t - c^2\sin^2 t}{\cos^2 t + c^2\sin^2 t} \tag{4}$$

   which is just the condition that $\boldsymbol{q} \in \boldsymbol{\Sigma}$ in (3).

4. Thus, every orbit is contained in a Clifford torus, and so the domain of the correspondence is foliated by invariant Clifford tori.

The explicit description of the dynamics allows us to determine when a Clifford torus admits periodic orbits. This already illustrates a difference compared to the symplectic counterpart, where non-degenerate odd-periodic orbits are guaranteed to exist if the table is a closed submanifold (Theorem 2 in [1]).

Proposition 3.1. *For almost all values of c, the outer contact billiard correspondence with respect to Σ_c admits no periodic orbits.*

Proof. Indeed, eliminating t from eqs. (4), periodic orbits on a Clifford torus correspond to values $c = (c_1/c_2)^2$ and $\alpha = q_1/p_1,\ \beta = q_2/p_2 \in \mathbb{Q}$ satisfying

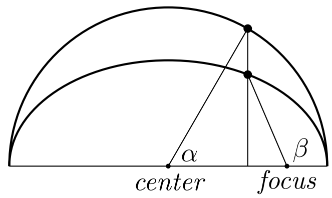
$$c^2 = \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)\left(\frac{1-\cos\beta}{1+\cos\beta}\right), \quad \text{or} \quad , \ c = \left|\frac{\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}}\right|.$$

As we let $q_j/p_j \in \mathbb{Q}$ range over the rationals, we have a countable number of parameter values, c, for Clifford tori Σ_c that admit periodic contact billiard orbits, while the remaining admit only quasi-periodic orbits. □

Remark 6. *The relation*

$$\frac{\tan^2 \frac{\beta}{2}}{\tan^2 \frac{\alpha}{2}} = c^2 \tag{5}$$

between the admissible angles α, β on a given Clifford torus with parameter $c = cst.$, is the same as the relation between the true and eccentric anomalies used to parametrize elliptic orbits of the Kepler problem. Geometrically, take a circle and ellipse (of eccentricity $e = \left|\frac{c^2-1}{c^2+1}\right|$) sharing a common diameter. Then, for $c^2 > 1$, the admissible angles are related via the figure:



whereas for $c^2 \in (0,1)$ the roles of α, β are interchanged.

It is natural to ask for the smallest possible period. As we will see in Section 4, this billiard correspondence does not admit 3-periodic orbits when $\dim V = 4$ (Proposition 4.1). However, the Clifford torus determined by the quadratic form

$$\frac{x^2 + y^2}{\sqrt{2} - 1} - \frac{u^2 + v^2}{\sqrt{2} + 1} = 0,$$

admits 4-periodic orbits. Figure 5 shows the periodic orbit with vertices:

$$(1 : 1 : 1 : 1),\ (0 : -1 : 0 : 1),\ (-1 : 1 : -1 : 1),\ (1 : 0 : -1 : 0).$$

Not only that, this is the only table defined by a quadratic form admitting 4-periodic orbits (see Proposition 4.2 below).

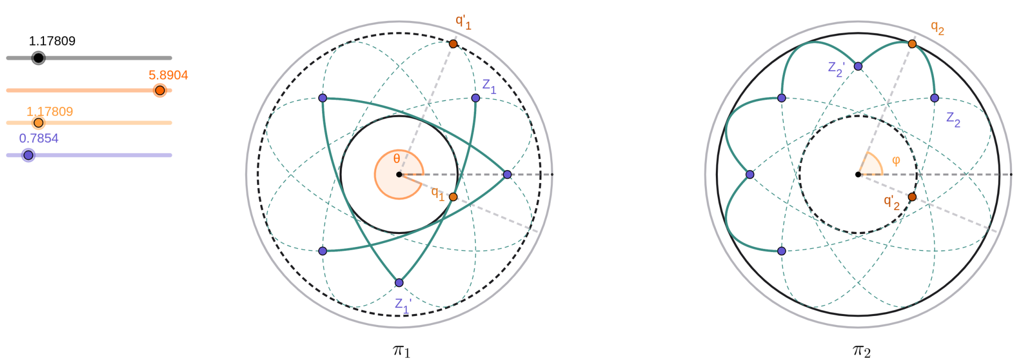


Figure 5: A 4-periodic orbit on the Clifford torus. Here, $\alpha = -3\pi/4, \beta = \pi/4$. After 4 iterations, the initial point (Z_1, Z_2) is its antipodal $(-Z_1, -Z_2)$.

Remark 7. *The projections of the characteristic lines onto each complex coordinate coincide with the trajectories of mechanical billiards (see e.g. [25]) along Hooke orbits (centered conics) bouncing off of certain centered circles.*

Summarizing, we have the following explicit description of the correspondence, as a special case of Proposition A.2.

Proposition 3.2. *The domain of the outer contact billiard map over* $\Sigma_c = \{x^2+y^2 = c(u^2+v^2)\} \subset \mathbb{RP}^3$*, with* $c > 0, c \neq 1$*, is the region foliated by the Clifford tori:*

$$I_{\sigma^2} = \left\{\sigma^2 = \frac{x^2 + y^2 - c(u^2 - v^2)}{c(u^2 + v^2 - c(x^2 + y^2))}\right\} \subset \mathbb{RP}^3$$

for $\sigma \in (0, \infty)$*. Each of these tori is invariant under the outer contact billiard map over* Σ_c*. The map is given, on* I_{σ^2} *by:*

$$I_{\sigma^2} \ni (x : y : u : v) \mapsto (x' : y' : u' : v') \in I_{\sigma^2}$$

where

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}.$$

The angles α, β *are constant on* I_{σ^2}*, determined by* $\sigma = \sqrt{\sigma^2} > 0$ *through:*

$$\frac{(1 - \sigma^2, 2\sigma)}{1 + \sigma^2} = (\cos\alpha, \sin\alpha), \quad \frac{(1 - c^2\sigma^2, -2c\sigma)}{1 + c^2\sigma^2} = (\cos\beta, \sin\beta)$$

and satisfying $c^2 = \frac{\tan^2\frac{\beta}{2}}{\tan^2\frac{\alpha}{2}}$.

3.2 Ellipsoids

Consider the table

$$E_a = \{a(x^2 + y^2) + u^2 = v^2\}, \quad a > 0,$$

which we see as an ellipsoid in the affine chart $\{v = 1\}$. In particular, this example does have some singular points which we are curious to examine. We find that the domain of the outer contact billiards map over E_a is filled by invariant ellipsoids, on each of which the dynamics is determined by iterating a certain fixed linear map (acting as a product of a usual rotation and a hyperbolic rotation). Namely (from Proposition A.2) we have explicitly:

Proposition 3.3. *The domain of the outer contact billiard map over the ellipsoid* $E_a = \{a(x^2 + y^2) + u^2 = v^2\} \subset \mathbb{RP}^3$*, with* $a > 0$*, is the region filled by the quadratics:*

$$I_{\sigma^2} = \left\{\sigma^2 = \frac{a(x^2 + y^2) + u^2 - v^2}{(x^2 + y^2)/a + v^2 - u^2}\right\} \subset \mathbb{RP}^3$$

for $\sigma \in (0, \infty)$*. The singular tangent planes of* E_a *constitute* I_{a^2}*. Each of these quadratic surfaces is invariant under the outer contact billiard map over* E_a*. When* $\sigma^2 \neq a^2$*, the map is given on* I_{σ^2} *by:*

$$I_{\sigma^2} \ni (x : y : u : v) \mapsto (x' : y' : u' : v') \in I_{\sigma^2}$$

for

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s & 0 & 0 \\ \sin s & \cos s & 0 & 0 \\ 0 & 0 & \epsilon\cosh t & \epsilon\sinh t \\ 0 & 0 & \epsilon\sinh t & \epsilon\cosh t \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix},$$

where $\epsilon = sgn\{a^2 - \sigma^2\}$*, and the angles* s, t *are constant on* I_{σ^2}*, determined through:*

$$\frac{(1-\sigma^2, 2\sigma)}{1+\sigma^2} = (\cos s, \sin s), \quad \frac{(a^2+\sigma^2, 2a\sigma)}{|a^2-\sigma^2|} = (\cosh t, \sinh t).$$

This table admits no periodic orbits.

This ellipsoid example exhibits exactly two singular points where $q_\pm = q'_\pm$, namely:

$$q_\pm = (0 : 0 : \pm 1 : 1), \qquad \xi_{q_\pm} = T_{q_\pm} E_a = \{u = \pm v\}.$$

The union of these singular tangent planes constituting the singular leaf $I_{a^2} = \xi_{q_+} \cup \xi_{q_-}$. Since, for points not on these singular planes, we have the billiard map explicitly, we can easily examine the limiting behavior induced on these singular planes. For this example, as we let a point limit to a position on one of these singular planes, we find well-defined limiting positions for its correspondents. Namely, tending to some point

$$Z_+ = (x_+ : y_+ : 1 : 1) \in \xi_{q_+} \backslash \{q_+\},$$

we find its 'regularized' correspondents are:

$$(0:0:1:1) = q_+ \in \xi_{q_+}, \text{ and } \left(2\frac{x_+\cos\alpha_* + y_+\sin\alpha_*}{(x_+^2+y_+^2)\sin\alpha_*} : 2\frac{-x_+\sin\alpha_* + y_+\cos\alpha_*}{(x_+^2+y_+^2)\sin\alpha_*} : -1 : 1\right) \in \xi_{q_-},$$

where $\tan\alpha_* = \frac{2a}{1-a^2}$. Likewise, a point $Z_- = (x_- : y_- : -1 : 1) \in \xi_{q_-} \backslash \{q_-\}$ has correspondents

$$(0:0:-1:1) = q_- \in \xi_{q_-}, \text{ and } \left(2\frac{x_-\cos\alpha_* - y_-\sin\alpha_*}{(x_-^2+y_-^2)\sin\alpha_*} : 2\frac{x_-\sin\alpha_* + y_-\cos\alpha_*}{(x_-^2+y_-^2)\sin\alpha_*} : 1 : 1\right) \in \xi_{q_+}$$

whereas a point on the intersection of the two singular planes has limiting correspondents the two singular points: $q_\pm$. We see then that one singular point acts as an attractor and the other as a repeller (which is also clear from the explicit description in Proposition 3.3). Otherwise, this limiting correspondence between the two singular planes is a rotation by the fixed angle α_* together with an inversion over a certain circle of radius $2/\sin\alpha_*$. We think that examining the behavior around singular points in general will be an interesting theme for future work (see Question 8 in Section 6 of open questions below).

Remark 8. *The pencil of quadratics* I_{σ^2} *form a singular foliation. Namely, each member of this family of surfaces is tangent at the singular points* $q_\pm$ *(see Figure 6).*

3.3 Sheared torus

Consider the table given by the quadric

$$\Sigma = \{v(v+x) = y(u-y)\},$$

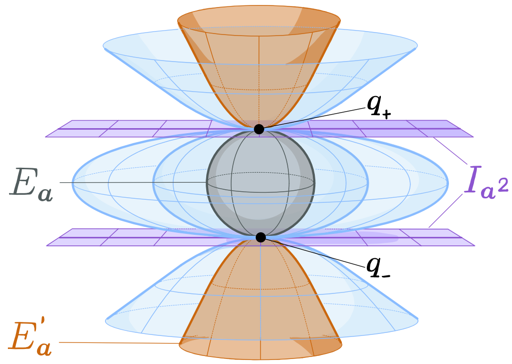


Figure 6: Singular foliation of invariant leaves on an affine chart. We show the pencil of quadratics I_{σ^2} and distinguish three special ones: the table, E_a, its polar surface E'_a and the union of the singular tangents I_{a^2}.

which is topologically a torus. Here the table and its polar intersect along a distinguished common line, ℓ_*. The outer billiard dynamics takes place along a family of invariant quadratic leaves, all intersecting along this same distinguished line. In this case, apart from collinear orbits contained in ℓ_*, there are no 'genuine' periodic orbits. Explicitly, the dynamics is given on each leaf by iterating a certain fixed linear map:

Proposition 3.4. *The domain of the outer contact billiard map over the torus $\Sigma = \{v(v+x) = y(u-y)\} \subset \mathbb{RP}^3$ is the region foliated by the quadrics*

$$I_{\sigma^2} = \left\{ \sigma^2 = \frac{y^2 + v^2 + xv - uy}{y^2 + v^2 - xv + uy} \right\} \subset \mathbb{RP}^3$$

for $\sigma \in (0, \infty)$. Each of these quadratic surfaces is invariant under the outer contact billiard map over Σ. On each leaf, the map is given by

$$I_{\sigma^2} \ni (x : y : u : v) \mapsto (x' : y' : u' : v') \in I_{\sigma^2}$$

as

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos s & \sin 2s & -\sin s & \cos 2s - 1 \\ 0 & \cos s & 0 & -\sin s \\ \sin s & 1 - \cos 2s & \cos s & \sin 2s \\ 0 & \sin s & 0 & \cos s \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix},$$

where the angle s is determined via

$$\frac{(1-\sigma^2, 2\sigma)}{1+\sigma^2} = (\cos s, \sin s)$$

and is constant on I_{σ^2}.

Proposition 3.5. *Apart from collinear orbits on the distinguished line $\ell_* = \{y = v = 0\}$, the table $\Sigma = \{v(v+x) = y(u-y)\}$ admits no periodic orbits.*

Proof. In the complex coordinates $\zeta = x + iu, \eta = y + iv$, the map is given by the (real) projectivization of:

$$(\zeta, \eta) \mapsto (\zeta', \eta') = e^{is}(\zeta + 2\sin s\ \eta, \eta).$$

Where for points not on Σ, we have $\eta \neq 0$ and $\sin s \neq 0$. If there were a periodic orbit, it would project to a periodic orbit under $\mathbb{C}^2 \to \mathbb{CP}^1$. But in the affine chart

$$(\zeta : \eta) \longleftrightarrow \zeta/\eta = z$$

this map is simply $z \mapsto z + 2\sin s$, and has no periodic orbits. $\square$

On this distinguished line, there are periodic orbits of any period: any two points are in contact correspondence. Generally, when we speak of genuine periodic orbits, we will have non-collinear ones in mind (see Remark 11).

3.4 Cylinders (pinched tori)

Now we will consider a degenerate quadratic surface:

$$\Sigma = \{x^2 + y^2 = v^2\} \subset \mathbb{RP}^3,$$

topologically a pinched torus with a singular point at $(0 : 0 : 1 : 0)$, which we see as a standard cylinder in the affine chart $\{v = 1\}$. For this table, the dynamics splits into a product of usual outer billiards in a plane together with a shift along the cylinder's axis. More explicitly:

Proposition 3.6. *For the table $\Sigma = \{x^2 + y^2 = v^2\}$, the outer contact billiard map is given in the affine chart $(x, y, u) \longleftrightarrow (x : y : u : 1)$ as follows: (x, y, u) and (x', y', u') are in correspondence over Σ when (see Figure 7)*

1. *(x, y) and (x', y') are in symplectic outer billiard correspondence over the unit circle and,*
2. *the vertical displacement $u' - u$ is the product pd of the distance d between $(x, y), (x', y')$, with the (signed) distance p from the origin to the line joining (x, y) and (x', y').*

Proof. The proof is essentially the same for a generalized cylinder, given by $f(x, y) = cst.$, in the affine chart $\{v = 1\}$, where the contact structure is $\xi|_{(x,y,u)} = \{du = xdy - ydx\}$. Note that the contact points of the vertical tangent planes lie at infinity, so that the harmonic conjugate condition is given in this affine chart by taking usual affine midpoints along the characteristic lines (see §5.1 for the more general situation). One computes that the characteristic lines along the generalized cylinder $\{f(x, y) = cst.\}$ are directed by:

$$\mathbf{v} = \frac{-f_y\partial_x + f_x\partial_y + (xf_x + yf_y)\partial_u}{\sqrt{f_x^2 + f_y^2}}$$

where $\frac{-f_y\partial_x + f_x\partial_y}{\sqrt{f_x^2+f_y^2}}$ directs the unit tangent to the level set $f(x, y) = cst.$, while $p = \frac{xf_x+yf_y}{\sqrt{f_x^2+f_y^2}}$ is the signed distance of this tangent to the origin so that $(x' - x, y' - y, u' - u) = \mathbf{v}d$ and the vertical shift $u' - u$ is as claimed. $\square$

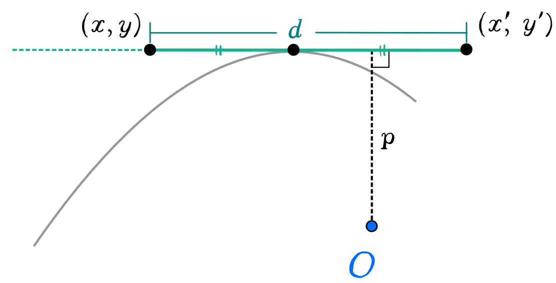


Figure 7: The points (x, y), (x', y') are in outer correspondence over $\{f(x, y) = cst\}$. The support function of the tangent line connecting $(x, y), (x', y')$ is p.

Remark 9. *Note that, in this case, the harmonic conjugate condition over this certain table has reduced in this affine chart to the usual notion of midpoint. We will see in Section 5.1 (Theorem 5.2) the reciprocal result: if the notion of harmonic conjugate and midpoint coincide in some affine chart, then the table must be a "generalized cone".*

Remark 10. *The vertical displacement along an orbit is determined by the net accumulation of the successive signed areas, pd, over the orbit. For a general closed cone, one can always choose an appropriate symplectic basis so that in the affine chart $\{v = 1\}$, the u-axis is contained in the interior of the corresponding generalized cylinder. In particular, the cumulative vertical displacements are then strictly increasing and there can be no periodic orbits. The singular point of the generalized cone acts as a repeller/attractor (as if one had glued together the singular poles on the ellipsoid example 3.2 above).*

4 Periodic orbits

The periodic trajectories in our quadratic examples above are surprisingly rare. This raises the question of which periods can occur for the outer contact billiard correspondence. In this section, we consider some basic properties of periodic outer contact billiards orbits when $\dim V = 4, \mathbb{P}V = \mathbb{RP}^3$.

Let $(Z_j)_{j\in\mathbb{Z}}$ be an orbit of the contact billiard correspondence. Every pair of consecutive points Z_j, Z_{j+1} is joined, by definition, by a characteristic tangent line, and is therefore contained in a Legendrian line of $(\mathbb{P}V, \xi)$ (see Figure 8). Consequently, every periodic orbit determines vertices of some closed Legendrian polygon (a discrete Legendrian knot). A related notion (equivalent when $\dim V = 4$) is explored in [10].

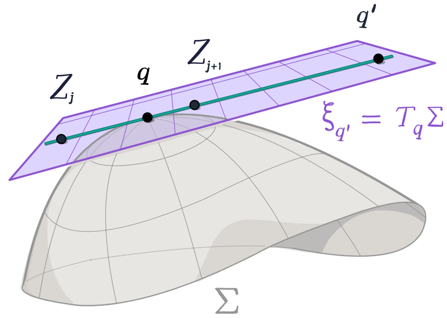


Figure 8: If $q \in Z_j \wedge Z_{j+1}$ is the reflecting point on the table with contact point q' then $\xi_{q'} \supset Z_j \wedge Z_{j+1} = q \wedge q' \ni q'$.

This geometric interpretation already imposes strong restrictions on the possible periods. In contrast to outer symplectic billiards, where 3-periodic orbits occur over any compact table (Theorem 2 of [1]), for outer contact billiards:

Proposition 4.1. *For any table $\Sigma \subset (\mathbb{RP}^3, \xi)$, the associated outer contact billiard correspondence admits no non-collinear 3-periodic orbits.*

Proof. A non-collinear 3-periodic outer contact billiard orbit would have Z_1, Z_2, Z_3 at the vertices of some Legendrian triangle in $(\mathbb{RP}^3, \xi)$. There are no Legendrian triangles here as the following argument shows: a Legendrian triangle would lift to an independent triple $\boldsymbol{Z_1}, \boldsymbol{Z_2}, \boldsymbol{Z_3} \in (\mathbb{R}^4, \omega)$, with $\omega(\boldsymbol{Z_1}, \boldsymbol{Z_2}) = \omega(\boldsymbol{Z_2}, \boldsymbol{Z_3}) = \omega(\boldsymbol{Z_3}, \boldsymbol{Z_1}) = 0$, and we would have a 3-dimensional isotropic subspace spanned by the $\boldsymbol{Z}_j$'s. But ω is non-degenerate, and its largest isotropic subspaces in $\mathbb{R}^4$ are the Lagrangian 2-planes. $\square$

Remark 11. *The non-collinearity assumption is necessary. Indeed, fix some Legendrian line $L \subset \xi_{q'}$ through a given point q' and choose three distinct points $Z_1, Z_2, Z_3 \in L \backslash q'$. For each pair (Z_j, Z_{j+1}), take the reflection point $q_j \in L$ as the harmonic conjugate of q' over Z_j, Z_{j+1}. Any smooth hypersurface tangent to $\xi_{q'}$ at the points $q_j \in L \subset \xi_{q'}$ admits, by construction, Z_1, Z_2, Z_3 as a collinear 3-periodic orbit. By the same construction, for any $n \in \mathbb{N}$, there exist smooth tables admitting collinear n-periodic orbits as well.*

On the other hand, non-degenerate 4-periodic outer contact billiards orbits do exist (as we have seen in Section 3.1). Let us briefly comment on the non-collinear Legendrian quadrilaterals in $(\mathbb{RP}^3, \xi)$, all of which are equivalent under projectivized symplectic transformations, $\mathrm{PSp}_4(\mathbb{R})$, i.e., in an appropriately adapted symplectic basis. Indeed, by Proposition 4.1, the vertices cannot lie in a projective plane, and therefore admit linearly independent lifts $\boldsymbol{Z_1}, \ldots, \boldsymbol{Z_4} \in \mathbb{R}^4$. As consecutive vertices are joined by Legendrian lines we have $\omega(\boldsymbol{Z_j}, \boldsymbol{Z_{j+1}}) = 0$ (where the subindices are read modulo four). After rescaling the lifts, we may further assume that $\omega(\boldsymbol{Z_1}, \boldsymbol{Z_3}) = \omega(\boldsymbol{Z_2}, \boldsymbol{Z_4}) = 1$, all other pairings being zero. In this symplectic basis, $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_4$, we have $\omega = dx \wedge dy + du \wedge dv$ for

$$Z_1 = \mathrm{span}\{\partial_x\},\ Z_2 = \mathrm{span}\{\partial_u\},\ Z_3 = \mathrm{span}\{\partial_y\},\ Z_4 = \mathrm{span}\{\partial_v\}.$$

To construct a table admitting this 4-periodic orbit, it suffices to choose reflection points $q_j \in Z_j \wedge Z_{j+1} \backslash \{Z_j, Z_{j+1}\}$ on each edge (see Figure 8). Such a choice determines the corresponding contact point q'_j as the harmonic conjugate of q_j with respect to Z_j, Z_{j+1}. The table Σ must contain the points q_j, and satisfy $T_{q_j}\Sigma = \xi_{q'_j}$. Among arbitrary smooth tables, satisfying these conditions presents no obstacle. On the other hand, the situation is more rigid within a prescribed family of tables where it is not immediate whether one can always fit such a table to the given 4-periodic orbit. For the family of quadratic tables, we find:

Proposition 4.2. *The* only *quadratic table in $\mathbb{RP}^3$ admitting a non-degenerate 4-periodic orbit is a Clifford torus given by:*

$$x^2 + y^2 = \frac{\sqrt{2}-1}{\sqrt{2}+1}(u^2 + v^2)$$

in appropriate symplectic coordinates ($\omega = dx \wedge dy + du \wedge dv$). It admits a 2-parameter family (invariant torus) of 4-periodic orbits (see Figure 5 above). For example:

$$(1 : 1 : 1 : 1),\ (0 : -1 : 0 : 1),\ (-1 : 1 : -1 : 1),\ (1 : 0 : -1 : 0).$$

Proof. Suppose that a quadratic table $\Sigma = \{Q = 0\}$, where Q is a quadratic form on $\mathbb{R}^4$, admits a non-collinear 4-periodic orbit. By the discussion preceding the proposition we may consider symplectic coordinates with

$$Z_1 = (1 : 0 : 0 : 0), Z_2 = (0 : 0 : 1 : 0), Z_3 = (0 : 1 : 0 : 0), Z_4 = (0 : 0 : 0 : 1),$$

and $\omega = dx \wedge dy + du \wedge dv$. Our general quadratic form is then given in these coordinates by

$$Q = Ax^2 + By^2 + Cu^2 + Dv^2 + 2\alpha yu + 2\beta xu + 2\gamma xy + 2axv + 2byv + 2cuv.$$

For the quadratic surface $\Sigma = \{Q = 0\}$ to admit this orbit, we require some reflecting points

$$q_j = \operatorname{span}\{\Lambda_j \boldsymbol{Z_j} + (1 - \Lambda_j)\boldsymbol{Z_{j+1}}\} \in \Sigma, \quad \Lambda_j \neq 0, 1$$

together with their harmonic conjugates

$$q'_j = \operatorname{span}\{\Lambda_j \boldsymbol{Z}_j + (\Lambda_j - 1)\boldsymbol{Z}_{j+1}\}$$

over Z_j, Z_{j+1} to satisfy $T_{q_j}\Sigma = \xi_{q'_j}$. These conditions are a system of equations where the unknowns are the coefficients of Q together with the four parameters Λ_j. At first sight, this appears to be an overdetermined problem: prescribing four points and four tangent hyperplanes on a quadratic typically imposes $4 \cdot 3 + 4 = 16$ conditions, whereas there are only $9 + 4 = 13$ unknowns (up to an overall scaling of Q). As it turns out, many of these conditions are redundant here because of the special configuration of our points and hyperplanes under consideration.

Recall that Q is homogeneous of degree two. For X_Q the symplectic gradient of Q then

$$\omega(\boldsymbol{x}, X_Q(\boldsymbol{x})) = d_{\boldsymbol{x}}Q(\boldsymbol{x}) = 2Q(\boldsymbol{x})$$

So that the tangency condition $T_q\Sigma = \xi_{q'}$, ie $\omega(\boldsymbol{q}, \boldsymbol{q}') = 0$ for $\boldsymbol{q}' = X_Q(\boldsymbol{q})$, already implies $q \in \Sigma$.

Therefore, it only remains to impose the tangency conditions $T_{q_j}\Sigma = \ker d_{q_j}Q = \xi_{q'_j}$. For $q = (x : y : u : v) \in \Sigma$, we compute its contact point, $\operatorname{span}\{X_Q(\boldsymbol{q})\}$, is at:

$$q' = (-(By + \alpha u + \gamma x + bv) : Ax + \beta u + \gamma y + av : -(Dv + ax + by + cu) : Cu + \alpha y + \beta x + cv).$$

So, upon setting

$$\Lambda'_j := 1 - \Lambda_j \neq 0, 1$$

the tangency conditions $T_{q_j}\Sigma = \xi_{q'_j}$, consist in the following system:

$$(\Lambda_1 : 0 : -\Lambda'_1 : 0) = (-(\alpha\Lambda'_1 + \gamma\Lambda_1) : A\Lambda_1 + \beta\Lambda'_1 : -(a\Lambda_1 + c\Lambda'_1) : C\Lambda'_1 + \beta\Lambda_1),$$
$$(0 : -\Lambda'_2 : \Lambda_2 : 0) = (-(B\Lambda'_2 + \alpha\Lambda_2) : \beta\Lambda_2 + \gamma\Lambda'_2 : -(b\Lambda'_2 + c\Lambda_2) : C\Lambda_2 + \alpha\Lambda'_2),$$
$$(0 : \Lambda_3 : 0 : -\Lambda'_3) = (-(B\Lambda_3 + b\Lambda'_3) : \gamma\Lambda_3 + a\Lambda'_3 : -(D\Lambda'_3 + b\Lambda_3) : \alpha\Lambda_3 + c\Lambda'_3),$$
$$(-\Lambda'_4 : 0 : 0 : \Lambda_4) = (-(\gamma\Lambda'_4 + b\Lambda_4) : A\Lambda'_4 + a\Lambda_4 : -(D\Lambda_4 + a\Lambda'_4) : \beta\Lambda'_4 + c\Lambda_4).$$

The system admits exactly two families of solutions:

1. $\Lambda_1\Lambda_2\Lambda_3\Lambda_4 = \Lambda'_1\Lambda'_2\Lambda'_3\Lambda'_4$, with

$$Q_+ = \left(x + \frac{\Lambda_1\Lambda_2}{\Lambda'_1\Lambda'_2}y - \frac{\Lambda_1}{\Lambda'_1}u - \frac{\Lambda'_4}{\Lambda_4}v\right)^2$$

2. $\Lambda_1\Lambda_3 = \Lambda'_1\Lambda'_3$, and, $\Lambda'_2\Lambda'_4 = -\Lambda_2\Lambda_4$, with

$$Q_- = x^2 + \left(\frac{\Lambda_1\Lambda_2}{\Lambda'_1\Lambda'_2}y\right)^2 + \left(\frac{\Lambda_1}{\Lambda'_1}u\right)^2 + \left(\frac{\Lambda'_4}{\Lambda_4}v\right)^2 - 2\frac{\Lambda_1}{\Lambda'_1}\left(\frac{\Lambda_1\Lambda_2}{\Lambda'_1\Lambda'_2}yu + ux\right) - 2\frac{\Lambda'_4}{\Lambda_4}\left(xv + \frac{\Lambda'_3\Lambda'_4}{\Lambda_3\Lambda_4}yv\right).$$

The first type of quadratic is degenerate: $Q_+ = 0$ just defines a hyperplane with no outer contact billiards dynamics. The second type of quadratic, $Q_- = 0$, upon diagonalizing is exactly the special Clifford torus stated above: set $\lambda_j = \Lambda_j/\Lambda'_j$ and consider the change of basis

$$X + U = \frac{x - \lambda_1\lambda_2 y}{\sqrt{2}}, \quad V + Y = \frac{x + \lambda_1\lambda_2 y}{\sqrt{2}}$$
$$V - Y = \lambda_1 u, \quad X - U = \lambda_2 v.$$

□

Remark 12. *From this last proof there is a 2-parameter family of quadratic surfaces admitting a given fixed Legendrian quadrilateral as a 4-periodic orbit. The family is parametrized by selecting any pair of points on adjacent sides of the Legendrian quadrilateral through which said surface must pass (the remaining pair of points through which the quadratic passes on the opposite sides are then determined uniquely through the conditions $\Lambda_1\Lambda_3 = \Lambda'_1\Lambda'_3$, and, $\Lambda'_2\Lambda'_4 = -\Lambda_2\Lambda_4$ in item 2 above).*

5 Preserved contact structure

In this section, we establish some general properties of the outer contact billiard map. Our first result shows that the name "contact billiards" is justified.

Theorem 5.1. *When a local diffeomorphism, the outer contact billiard map is a contactomorphism.*

That is, given a regular hypersurface $\Sigma \subset (\mathbb{P}V, \xi)$, the outer contact billiards correspondence over Σ (Definition 2.3) defines a locus in $\mathbb{P}V \times \mathbb{P}V$ consisting of those pairs in outer contact billiards correspondence over Σ. When smooth and transverse to the fibers, this locus is given locally by the graph of a contactomorphism (see Remark 13 below).

Proof. Let $\Sigma \subset (\mathbb{P}V, \xi)$ be a regular hypersurface where ξ is a standard contact structure on $\mathbb{P}V$ induced by a linear symplectic structure (V, ω). Suppose that the outer contact billiard correspondence determined by Σ is a local diffeomorphism on an appropriate domain, and consider a smooth curve of tuples

$$t \mapsto (Z(t), q(t), Z'(t), q'(t))$$

in outer contact billiards correspondence over Σ. Since Z, Z' are harmonic conjugates over q, q', we may choose lifts $\boldsymbol{Z}, \boldsymbol{q}, \boldsymbol{Z'}, \boldsymbol{q'} \in V$ such that

$$\boldsymbol{Z} = \boldsymbol{q} + \boldsymbol{q'}, \quad \boldsymbol{Z'} = \boldsymbol{q} - \boldsymbol{q'}.$$

Write $\dot{\boldsymbol{Z}}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{Z'}}$, and $\dot{\boldsymbol{q'}}$ for the corresponding derivatives.

Since $q \in T_q\Sigma = \xi_{q'}$, we have

$$\omega(\boldsymbol{q}, \boldsymbol{q'}) = 0 \implies \omega(\dot{\boldsymbol{q}}, \boldsymbol{q'}) = \omega(\dot{\boldsymbol{q'}}, \boldsymbol{q}).$$

In fact, since $\dot{q} \in T_q\Sigma = \xi_{q'}$ we have:

$$\omega(\dot{\boldsymbol{q}}, \boldsymbol{q'}) = 0 = \omega(\dot{\boldsymbol{q'}}, \boldsymbol{q}). \tag{6}$$

Using Equation 6, we compute

$$\omega(\boldsymbol{Z}, \dot{\boldsymbol{Z}}) = \omega(\boldsymbol{q} + \boldsymbol{q'}, \dot{\boldsymbol{q}} + \dot{\boldsymbol{q'}}) = \omega(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \omega(\boldsymbol{q'}, \dot{\boldsymbol{q'}}) = \omega(\boldsymbol{q} - \boldsymbol{q'}, \dot{\boldsymbol{q}} - \dot{\boldsymbol{q'}}) = \omega(\boldsymbol{Z'}, \dot{\boldsymbol{Z'}}),$$

and, in particular:

$$\dot{Z} \in \xi_Z \iff \omega(\boldsymbol{Z}, \dot{\boldsymbol{Z}}) = 0 = \omega(\boldsymbol{Z'}, \dot{\boldsymbol{Z'}}) \iff \dot{Z}' \in \xi_{Z'}.$$

□

Examining the proof more closely, one finds a stronger statement: in a certain sense, the outer contact billiards map (Definition 2.3) is the *only* such reflection rule generating contact transformations over any table.

Corollary 5.1. *Consider a non-trivial outer billiard reflection rule on $(\mathbb{P}V, \xi)$ which reflects Z to Z' over characteristic lines of tables $\Sigma \subset \mathbb{P}V$. Then the induced maps are local contact transformations if and only if they are given by the reflection rule described in Definition 2.3 above.*

Proof. Using the notation of the previous proof, suppose the reflection rule is given by $\boldsymbol{Z'} = \boldsymbol{q} + \beta\boldsymbol{q'}$, where the coefficient β is given by some function determining the "twisted" reflection rule. Then we have: $\omega(\boldsymbol{Z'}, \dot{\boldsymbol{Z'}}) = \omega(\boldsymbol{Z}, \dot{\boldsymbol{Z}}) + (\beta^2 - 1)\omega(\boldsymbol{q'}, \dot{\boldsymbol{q'}})$. We generate contact transformations over arbitrary tables when $0 = \omega(\boldsymbol{Z}, \dot{\boldsymbol{Z}}) \iff 0 = \omega(\boldsymbol{Z'}, \dot{\boldsymbol{Z'}})$. That is, when $0 = (\beta^2 - 1)\omega(\boldsymbol{q'}, \dot{\boldsymbol{q'}})$. Since we are free to move on a general table, $\omega(\boldsymbol{q'}, \dot{\boldsymbol{q'}})$ can be arbitrary, and the only such choice of reflection rule is with $\beta = \pm 1$, i.e. either that of Definition 2.3 (when $\beta = -1$) or the trivial reflection rule, $Z = Z'$ (when $\beta = 1$). □

Remark 13. *Given two contact manifolds $(N, \xi), (N', \xi')$ their product inherits a natural pair of hyperplane distributions $\pi^*\xi = \{v \in T(N \times N') : \pi_* v \in \xi\}$ and $(\pi')^*(\xi')$ for π, π' the standard projections onto each factor. A submanifold $\Gamma \subset N \times N'$ such that $(T\Gamma) \cap (\pi^*\xi) = (T\Gamma) \cap ((\pi')^*\xi')$ is, when transverse to the fibers, given locally by the graph of a contactomorphism.*

5.1 Affine tables

We note that harmonic conjugate reduces to taking midpoints in an affine chart when one of the points in the tuple is a point at infinity. Our outer contact billiards takes place in projective space, but naturally we would like to view it also in some affine chart. In such an affine chart, there appears a less complicated reflection rule since we now have a notion of midpoints.

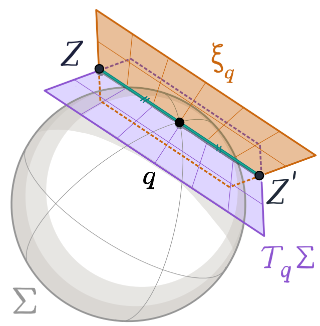


Figure 9: The midpoint reflection rule in an affine chart: $q \in \Sigma$ is the midpoint of Z, Z'

Typically our outer contact billiards reflection and this midpoint rule will *not* coincide, however we may still ask: for which tables do these two rules coincide? When could this more familiar midpoint reflection rule generate contact transformations?

As it turns out, such tables must be of the type considered in our examples from §3.4 above:

Theorem 5.2. *Let $\Pi_\infty \in \mathbb{P}V^*$ be a fixed hyperplane as the plane at infinity of an affine chart. Then the following are equivalent:*

1. *the midpoint reflection rule over a table $\Sigma \subset \mathbb{P}V$ in this affine chart generates a contact transformation of $(\mathbb{P}V, \xi)$,*
2. *the midpoint reflection rule over Σ coincides with the outer contact billiards reflection rule over Σ (Definition 2.3),*
3. *Σ is a generalized cone with vertex at the contact point $q_\infty \in \Pi_\infty$ of the plane at infinity (the point where $\Pi_\infty = \xi_{q_\infty}$).*

Proof. The equivalence of (1) $\iff$ (2) follows from Corollary 5.1.

To see the equivalence (2) $\iff$ (3), we note that the midpoint reflection rule is precisely harmonic conjugation with respect to a point at infinity. Therefore, the midpoint rule coincides with that of Definition 2.3 if and only if, for every $q \in \Sigma$, the contact point $q' = (T_q\Sigma)^\omega$ lies on the hyperplane at infinity Π_∞, or equivalently, $(T_q\Sigma)^\omega \subset \Pi_\infty$. By taking symplectic complements we have $T_q\Sigma \supset (\Pi_\infty)^\omega = q_\infty$ so that such planes are exactly those belonging to the pencil of hyperplanes incident to q_∞. Hence, every tangent hyperplane to Σ passes through the fixed point q_∞. These are precisely the generalized cones in item (3), see Figure 10.

□

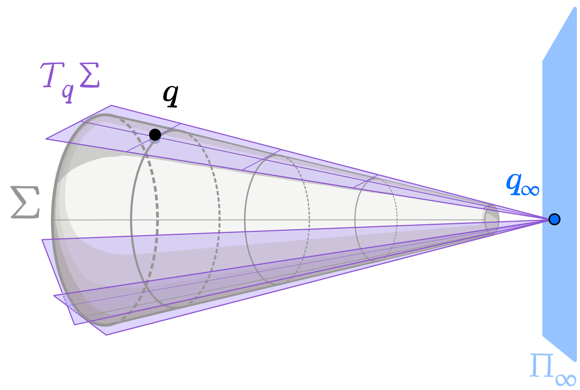


Figure 10: Generalized cone with vertex at infinity.

5.2 Hopf tables

It is also standard when studying the projective space to pass to a double cover, by considering oriented lines (rays), and making some choice of inner product in order to work then on an appropriate unit sphere. Similarly to our affine chart of the last section, after such choices there now appears a simpler reflection rule, by using the inner product to reflect with equal angles along characteristic lines.

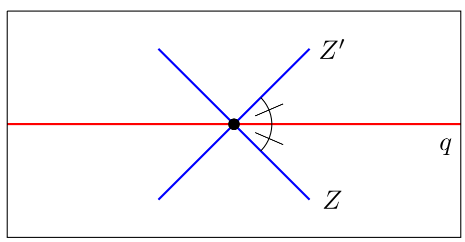


Figure 11: The angle reflection rule with respect to an inner product: $\angle(q, Z) = \angle(q, Z')$.

Typically our outer contact billiards reflection and this angle reflection rule will *not* coincide, however we may ask again: for which tables do these two rules coincide? When could this more familiar equal angles reflection rule generate contact transformations?

As it turns out such tables must be of the type considered in our example from §3.1 above:

Theorem 5.3. *Let $J : V \to V$ be some ω-compatible complex structure, $J^*\omega = \omega$, with associated inner product $g_J(u, v) = \omega(u, Jv)$. Then the following are equivalent:*

1. *the angle reflection rule over a table $\Sigma \subset \mathbb{P}V$ with respect to the inner product g_J generates a contact transformation of $(\mathbb{P}V, \xi)$,*

2. *the angle reflection rule over Σ with respect to g_J coincides with the outer contact billiards reflection rule over Σ (Definition 2.3),*

3. Σ *is a* Hopf table*, that is to say,* $\Sigma = \pi_J^{-1}(\Gamma)$ *for some hypersurface* $\Gamma \subset \mathbb{CP}^{n-1}$ *and* $\pi_J : V \to \mathbb{P}_{\mathbb{C}}V \cong \mathbb{CP}^{n-1}$ *the Hopf map associated to the complex structure* J *on* $V \cong \mathbb{C}^n$*.*

Proof. The equivalence of (1) $\iff$ (2) is Corollary 5.1.
To see the equivalence between (2) and (3), let $q \in \mathbb{P}V$ be a line through the origin, and $\Pi \in \mathbb{P}V^*$ a hyperplane containing q so that

$$\Pi \supset q \implies \xi_q = q^\omega \supset \Pi^\omega$$

and, when $\Pi \neq q^\omega$, the characteristic line of $\Pi \ni q$ is the 2-plane spanned by q and Π^ω. The equal angles reflection rule with respect to g_J will coincide with the outer contact billiard reflection rule over Π if and only if q and Π^ω are g_J orthogonal subspaces (see Figure 12).

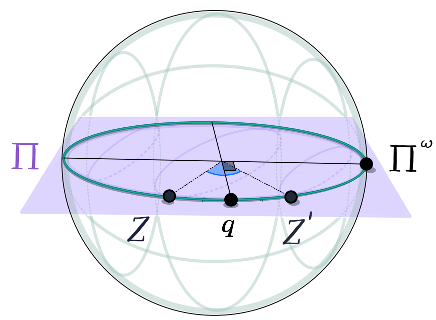


Figure 12: Harmonic conjugates and equal angles coincide when the reflecting point $q \in \Pi$ and Π^ω are perpendicular.

That is, for $\Pi^\omega = \text{span}\{u\}$, we have:

$$0 = g(u, q) = \omega(u, Jq) \iff Jq \in (\Pi^\omega)^\omega = \Pi.$$

Applied to the tangent planes $\Pi = T_q\Sigma$ of some table Σ, we see that the equal angles reflection rule over the table coincides with the outer contact billiard reflection rule if and only if $Jq \in T_q\Sigma$. Since Jq is the infinitesimal generator of the S^1-action $q \mapsto e^{it}q$, the condition $Jq \in T_q\Sigma$ means that Σ is invariant under this action. Hence Σ is a union of fibers of the associated Hopf fibration. $\square$

Remark 14. *In this last result, it is not necessary that the associated inner product* g_J *be definite, so that midpoint reflection rules in hyperbolic spaces can also be included.*

6 Open questions

The theory developed here leaves many further questions to be explored. Some concern the geometric foundations of outer contact billiards, while others point towards possible extensions and connections with classical billiard theory. We collect a few of these problems below.

1. **Local contactomorphisms as contact billiards.** Theorem 5.1 shows that when the outer contact billiard correspondence is a local diffeomorphism, the map preserves the contact structure. Can every local contactomorphism be realized as an outer contact billiard map?

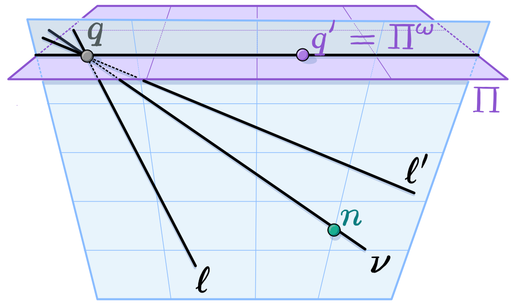


Figure 13: The proposed reflection law defined using only projective and contact structure. Here the line ν plays the same role as the transversal line field in projective billiards (see [23]). However, ν depends not only on the reflecting plane Π and incidence point q, but also on the incident line ℓ (this normal remaining fixed among incident lines in the pencil through $\ell, q \wedge q'$).

2. **Inner contact billiards.** Throughout this paper, we focus on *outer* billiards. Is there a natural *inner* contact billiard? What kind of structure should these preserve?

   For instance, let us consider $(\mathbb{RP}^3, \xi)$ where it is possible to cook up various candidate reflection laws for incident lines to a given surface. The reflection law off of said surface then induces a certain local transformation of $\mathrm{Gr}(2, \mathbb{R}^4)$. What is an appropriate analogue of Theorem 5.1 to decide which reflection law shall earn the name of inner contact billiards?

   For example, a candidate reflection law in $(\mathbb{RP}^3, \xi)$ can be defined (using only the projective and contact structure) in the following way:

   - Given a line ℓ incident to a hyperplane Π at $q \in \Pi \backslash \{q'\}$ (for $q' = \Pi^\omega$ the contact point of Π).
   - Let n be the contact point of the plane through q' and ℓ, and $\nu = q \wedge n$ the line through q and n.
   - Then consider ℓ' to be the reflection of ℓ off of Π when:
     (a) ℓ' lies in the pencil of lines generated by ℓ, ν, and
     (b) ℓ, ℓ' are harmonic conjugates with respect to $q \wedge q', \nu$.

   See Figure 13.

3. **Variational principle.** The outer contact billiard construction is defined simultaneously from a contact and a projective structure. Is there a natural variational principle associated with this geometry? More specifically, can outer contact billiard trajectories be characterized as critical points of a suitable action functional in a similar way that Herglotz' variational principle characterizes trajectories of contact vector fields? See for example [7], where a discrete Herglotz variational principle (generating contact transformations) has been developed.

4. **Projective duality.** Is there a natural dual construction for outer contact billiards under projective duality?

5. **Higher-dimensional quadrics.** Our explicit description of outer contact billiards over quadratic tables was restricted to $\mathbb{RP}^3$. Is the outer contact billiard map over quadratic surfaces still integrable in higher dimensions? See Remark 18 in the Appendix below.

6. **Higher degree tables.** The contact billiards over quadratic tables in $(\mathbb{RP}^3, \xi)$ has been described here completely. What can one say about higher degree tables?

   Some comments:

   - The characteristic lines along a general surface in $\mathbb{RP}^3$ form a 2-parameter family of lines, or *congruence* of lines, in $\mathbb{RP}^3$ (see [11]: Book IV).
   - For quadratic surfaces in $\mathbb{RP}^3$, their outer contact billiards induce a bona fide dynamics: through each point there pass at most *two* characteristic lines (one a 'future' and the other a 'past'). For higher degree tables, this need not be so. Indeed, for a degree d surface in $\mathbb{RP}^3$ typically there will pass $d(d-1)$ characteristic lines through a given point (the congruence of characteristic lines has *order* $d(d-1)$ in the terminology of [11]), and one will need to consider a type of 'branched dynamics' as we have seen in remark 18.
   - A given congruence of lines generally envelope a certain focal surface, which may be composed of various branches. A given line may touch the focal surface at multiple points (each point along some branch). Apart from the table and its polar, what sorts of focal surfaces are associated with a congruence of characteristic lines along an algebraic table in $(\mathbb{RP}^3, \xi)$? What sorts of surface transformations and sequences of conjugate nets (see §5 of [9]) can arise from such congruences composed of characteristic lines?

7. **Higher-codimension tables.** The outer contact billiard construction admits a natural extension beyond tables that are hypersurfaces (codimension one). For a submanifold of codimension greater than one, each tangent space determines a family of characteristic lines, rather than a single one, and we have still an induced outer contact billiards correspondence. What new phenomena emerge in this higher- codimensional setting?

8. **Singular points.** Singular points arise in many classes of projective tables, yet their role in the dynamics requires further investigation (see §3.2). What sort of behaviors can appear around these singular points? When is it possible to regularize the correspondence through such singularities?

9. **Relations to dissipative billiards.** Contact geometry provides a broad framework for dissipative dynamics, and so the outer contact billiards here can be considered as a certain type of dissipative billiard (we see this dissipation in several of our examples already, e.g. the attracting and repelling points in Example 3.2).

   Dissipative billiards are usually considered as certain conformally symplectic transformations in their corresponding phase space (see [14], [5], and [4]). Conformal symplectic transformations are special cases of contact transformations, lifting, locally at least, to contact transformations of an appropriate contactification. Thus one can imagine that these usual dissipative billiards should correspond to an appropriate projection of contact billiards. Can contact billiards help understand these usual dissipative billiards?

   Separately, one can mimic the definition of dissipative symplectic billiards in the context of contact billiards. Instead of asking for the cross ratio in Definition 2.3 to be -1, one can replace it with a different fixed value $\beta \neq -1$. By Corollary 5.1, we know such billiards do not preserve the contact structure. Is there any preserved structure?

10. **Classical billiard conjectures.** In the case of Birkhoff (Euclidean) billiards, there are two "holy grails": the Ivrii and Birkhoff-Poritsky conjectures. One can also pose the analogs of these conjectures in this new setting:
    - (*Ivrii* [13]) Does the set of periodic contact billiard orbits have measure zero?
    - (*Birkhoff-Poritsky* [18]) Given a hypersurface Σ such that the outer contact billiard correspondence is integrable, can we ensure Σ is a quadratic table?

11. **Periodic Clifford tori.** We showed that the dynamics on Clifford tori, Section 3.1, are completely determined by two rotation frequencies satisfying an explicit relation (Equation (5)). For a given Clifford torus, which periods can actually occur?

A Quadratic tables

Our examples above, §3, concern quadratic tables (in $(\mathbb{RP}^3, \xi)$). The space of quadratic forms on a symplectic vector space is well studied (see [26], or §2.4 of [3]). For completeness, we recall here that modulo symplectic changes of basis there is a 2-parameter family of quadratic forms on $(\mathbb{R}^4, \omega)$, in correspondence with the adjoint orbits of $\mathfrak{sp}_4(\mathbb{R})$. Consequently, modulo re-scalings, we have a 1-parameter family of quadratic tables in $(\mathbb{RP}^3, \xi)$. Explicitly:

Proposition A.1 (Special case of [26])**.** *A non-trivial (i.e. non-empty, and non-planar) quadratic surface in $(\mathbb{RP}^3, \xi)$ is given, in an appropriate symplectic basis $(x, y, u, v) \in \mathbb{R}^4$ with $\omega = dx \wedge dy + du \wedge dv$ by one of the following normal forms:*

1. $\{x^2 + y^2 = v^2\}, \{xy = v^2\},\ \{vy = x^2\}$,
2. $\{cxy = uv\}$
3. $\{c(x^2 + y^2) = u^2 + v^2\}$,
4. $\{c(x^2 + y^2) + u^2 = v^2\}$,
5. $\{x(cy + v) = u(y - cv)\}$,
6. $\{xy + uv = uy\}$,
7. $\{v(v + x) = y(u - y)\}$

for $c \in \mathbb{R}\backslash 0$ a parameter.

Proof. We summarize a derivation. Consider our non-degenerate symplectic form as an invertible skew-linear map $\omega : \mathbb{R}^4 \to \mathbb{R}^{4*}$. Given a quadratic form, $Q : \mathbb{R}^4 \to \mathbb{R}$, let $S : \mathbb{R}^4 \to \mathbb{R}^{4*}$ be the (symmetric) map for Q's underlying inner product: $Q(\mathbf{v}) = (S\mathbf{v}, \mathbf{v})$. Then:

$$X = \omega^{-1} S \in \mathfrak{sp}_4(\mathbb{R})$$

(the linear vector field $\mathbf{v} \mapsto X\mathbf{v}$ is, up to constant multiples, the symplectic gradient of the quadratic Hamiltonian $\mathbf{v} \mapsto (S\mathbf{v}, \mathbf{v})$). Classifying quadratic forms up to symplectic changes of basis: $S \mapsto$

A^*SA corresponds to classifying $X \in \mathfrak{sp}_4(\mathbb{R})$ up to adjoint action, $X \mapsto A^{-1}XA$, for $A \in \mathrm{Sp}_4(\mathbb{R})$: $A^*\omega A = \omega$. These adjoint orbits are distinguished by the various types of eigenspaces of $X \in \mathfrak{sp}_4(\mathbb{R})$ (recall: if λ is an eigenvalue of $X \in \mathfrak{sp}_4(\mathbb{R})$, so too is $-\lambda, \bar{\lambda}, -\bar{\lambda}$). The items above correspond to the following cases 1: degenerate quadratic forms (some zero eigenvalues); 2: simple real eigenvalues $(a, -a, b, -b)$; 3: simple pure imaginary eigenvalues $(\pm ia, \pm ib)$; 4: simple real and pure imaginary eigenvalues $(a, -a, \pm ib)$; 5: general complex eigenvalue $(a \pm ib, -a \pm ib)$; 6: repeated real eigenvalue with a Jordan block; 7: repeated pure imaginary eigenvalue with a Jordan block. Finally, we use the re-scaling freedom to normalize some (non-zero) eigenvalues to one. $\square$

Remark 15. *For a vector space V we denote the natural pairing between $\nu \in V^*, v \in V$ by (ν, v).*

Remark 16. *In symplectic coordinates upstairs, $\omega = dx\wedge dy + du\wedge dv$, the induced contact structure is given by (see e.g. [6]) the kernel of the restriction of*

$$\lambda = \iota_{x\partial_x + y\partial_y + u\partial_u + v\partial_v}\omega = -ydx + xdy - vdu + udv$$

to any surface transversal to the radial direction, e.g. restriction of λ to an affine hyperplane.

Remark 17. *The cases with degenerate quadratic forms (item 1) are considered in example 3.4 above. Item 2 in the list corresponds to the quadratic table described in Proposition A.3. Item 3 corresponds to the Clifford tori exposed in example 3.1. Item 4 are the ellipsoids, considered in example 3.2. Item 5 corresponds to the quadratic case discussed in Proposition A.4. Item 6 is represented in Proposition A.5. Finally, item 7 of the list is precisely the case exposed in example 3.3.*

Continuing with the notation from the last proof, we compute the following description of outer contact billiards over non-degenerate quadratic surfaces:

Proposition A.2. *Let $\Sigma = \{\mathbf{v} : (S\mathbf{v}, \mathbf{v}) = 0\} \subset \mathbb{RP}^3$ be a non-degenerate quadratic surface with generator*

$$X = \omega^{-1}S \in \mathfrak{sp}_4(\mathbb{R}).$$

Then the polar surface of Σ with respect to ω is the quadratic surface $\Sigma' = \{\mathbf{v} : (S'\mathbf{v}, \mathbf{v}) = 0\} \subset \mathbb{RP}^3$ for $X^*S'X = S$, or:

$$S' = -\omega S^{-1}\omega.$$

The domain of the outer contact billiards map over Σ (or Σ') is the region foliated by the quadratics

$$I_{\sigma^2} = \{\mathbf{v} : (S\mathbf{v}, \mathbf{v}) = \sigma^2 \det X \ (S'\mathbf{v}, \mathbf{v})\}, \quad \sigma \in (0, \infty)$$

from the pencil of quadratics through Σ, Σ'. Each quadratic I_{σ^2} is invariant under the outer contact billiards map over Σ (or Σ'), explicitly by iterating

$$I_{\sigma^2} \ni span\{\boldsymbol{Z}\} \mapsto span\{(id - \sigma X)(id + \sigma X)^{-1}\boldsymbol{Z}\} \in I_{\sigma^2}$$

as long as $1/\sigma$ is not an eigenvalue of X.

Proof. The polar surface of Σ is the locus $\Sigma' = \{\mathrm{span}\{X\boldsymbol{q}\} : \mathrm{span}\{\boldsymbol{q}\} \in \Sigma\}$ so that S' is as claimed. A point $Z = \mathrm{span}\{\boldsymbol{Z}\}$ is in outer contact correspondence with Z' over Σ if and only if

$$\boldsymbol{Z} = \boldsymbol{q} + \sigma\boldsymbol{q'} = (id + \sigma X)\boldsymbol{q}$$

for some $q = \text{span}\{\boldsymbol{q}\} \in \Sigma$ and $\sigma \in \mathbb{R}$, in which case $Z' = \text{span}\{(id - \sigma X)\boldsymbol{q}\}$. It holds that $q \in \Sigma$ if and only if:

$$(S\boldsymbol{Z}, \boldsymbol{Z}) = \sigma^2(S\boldsymbol{q'}, \boldsymbol{q'}) = \sigma^2\omega(X^3\boldsymbol{q}, \boldsymbol{q}), \quad (S'\boldsymbol{Z}, \boldsymbol{Z}) = (S'\boldsymbol{q}, \boldsymbol{q}) = \omega(\boldsymbol{q}, X^{-1}\boldsymbol{q}).$$

Likewise, we compute $(S\boldsymbol{Z'}, \boldsymbol{Z'}) = \sigma^2(S\boldsymbol{q'}, \boldsymbol{q'}) = (S\boldsymbol{Z}, \boldsymbol{Z})$ and $(S'\boldsymbol{Z'}, \boldsymbol{Z'}) = (S'\boldsymbol{q}, \boldsymbol{q}) = (S'\boldsymbol{Z}, \boldsymbol{Z}))$. Thus the outer contact billiards over a quadratic table admit the integral:

$$I(x) = \frac{(S\mathbf{x}, \mathbf{x})}{(S'\mathbf{x}, \mathbf{x})}, \quad x = \text{span}\{\mathbf{x}\} \in \mathbb{RP}^3$$

or, in other words, the pencil of quadratics through Σ, Σ' are invariant. To describe the dynamics on each leaf, we restrict to the low-dimensional case. Computing in the basis $X^{-1}\boldsymbol{q}, \boldsymbol{q}, X\boldsymbol{q}, X^2\boldsymbol{q}$ we see

$$\omega(X^3\boldsymbol{q}, \boldsymbol{q}) = \det X \ \omega(\boldsymbol{q}, X^{-1}\boldsymbol{q})$$

So that $I = \sigma^2 \det X = cst.$ (for fixed $X \in \mathfrak{sp}_4(\mathbb{R})$) and the dynamics on this leaf is by iterating the fixed projective transformation stated above ($\sigma = cst.$). $\square$

Remark 18. *Our explicit description above of the dynamics on the pencil of invariant quadratics is special to the low-dimensional case* $(\mathbb{RP}^3, \xi)$*. In arbitrary dimensions we still have the invariant pencil of quadratics through* Σ, Σ'*:*

$$cst. = I(Z) = \frac{(S\boldsymbol{Z}, \boldsymbol{Z})}{(S'\boldsymbol{Z}, \boldsymbol{Z})} = I(Z').$$

However, the relation between the integral I to the dynamics on each leaf (i.e. the parameter σ above) may differ from that in Proposition A.2, where we had simply $cst. = \det X = \frac{I}{\sigma^2}$.

Namely, outer contact billiards over quadratics in higher dimensions will generally admit more integrals. Moreover, on each leaf (i.e. fixed level of the integrals), there will not typically be a traditional dynamics but a 'branched dynamics', where each point is in contact correspondence with more *than just* two *(past/future) points.*

We can already see this difference for the quadratic hypersurface

$$\Sigma = \{xu + ayv + bzw = 0\} \subset \mathbb{RP}^5, \quad \omega = du \wedge dx + dv \wedge dy + dw \wedge dz.$$

Through a general point in $\mathbb{RP}^5$ there pass up to four *characteristic lines of Σ, and the outer contact billiards over Σ admit* two *independent integrals. More explicitly, the outer contact billiards over Σ admits the general integral I from Proposition A.2 given by:*

$$I = \frac{xu + ayv + bzw}{xu + \frac{yv}{a} + \frac{zw}{b}}.$$

However, there are more integrals; in fact, we have individually the integrals:

$$I_1 = \frac{yv}{xu}, \quad I_2 = \frac{zw}{xu}$$

so that $I = \frac{1+aI_1+bI_2}{1+I_1/a+I_2/b}$. On each leaf $I_1 = cst., I_2 = cst.$, the correspondents are given through

$$(x : u : y : v : z : w) \longleftrightarrow \left(\frac{1-\sigma}{1+\sigma}x : \frac{1+\sigma}{1-\sigma}u : \frac{1-a\sigma}{1+a\sigma}y : \frac{1+a\sigma}{1-a\sigma}v : \frac{1-b\sigma}{1+b\sigma}z : \frac{1+b\sigma}{1-b\sigma}w\right).$$

where σ may take up to four values and, modulo signs, up to two *values. Namely, any root of:*

$$a^2b^2(\sigma)^4 - \left(\frac{(a^2+b^2)+a(1+b^2)I_1+b(1+a^2)I_2}{1+\frac{I_1}{a}+\frac{I_2}{b}}\right)(\sigma)^2 + I = 0.$$

So in higher dimensions, already for quadratic hypersurfaces, the variety of possibilities to describe has become much richer.

A.1 Explicit formulas

For completeness, we state here explicit formulas for the outer contact billiards induced by the remaining types of quadratic tables in $(\mathbb{RP}^3, \xi)$, only noting that such tables are also integrable in the same sense as the previous examples of quadratic tables.

For item 2 from Proposition A.1:

Proposition A.3. *The domain of the outer contact billiard map over the hyperbolic torus $\Sigma_c = \{cxy = uv\} \subset \mathbb{RP}^3$, with $c > 0, c \neq 1$, is the region foliated by the quadratics:*

$$I_{\sigma^2} = \left\{\sigma^2 = \frac{uv - cxy}{c(cuv - xy)}\right\} \subset \mathbb{RP}^3,$$

where $\sigma \in (0, \infty)$. The singular tangent planes of Σ_c comprise the degenerate quadratics $I_1, I_{1/c^2}$. Each one of these quadratic surfaces is invariant under the outer contact billiard map over Σ_c. When $\sigma^2 \neq 1, 1/c^2$, the map is given, on I_{σ^2} by:

$$I_{\sigma^2} \ni (x : y : u : v) \mapsto (x' : y' : u' : v') \in I_{\sigma^2}$$

with

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_2} \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix},$$

where $\lambda_1 = \frac{1+c\sigma}{1-c\sigma}, \lambda_2 = \frac{1-\sigma}{1+\sigma}$, for $\sigma = \sqrt{\sigma^2} > 0$, are constant on I_{σ^2}. Note that since $\sigma \neq 0, 1, 1/c$, we have $\lambda_j \in \mathbb{R}\backslash\{0, \pm 1\}$. In particular, this implies that Σ_c has no periodic orbits.

Remark 19. *Here the case when $c = 1$ is exceptional: we have a 'self-dual' torus ruled by characteristic lines. There are two ruling lines which are singular (and transversal to the characteristic ruling lines). In this case, the limiting behavior on these singular lines is not completely trivial; it is a 2-periodic map.*

Remark 20. *Although Proposition A.3 describes the domain as foliated by invariant hyperbolic tori, the closure of these leaves is not disjoint. Any two leaves intersect along four projective lines*

$$(x : 0 : u : 0),\ (x : 0 : 0 : v),\ (0 : y : u : 0),\ (0 : y : 0 : v),$$

all contained in the table Σ_c. Since these lines are entirely inside the billiard table, removing Σ_c, Σ'_c leaves a genuine foliation of the billiard domain.

Next, item 5 from Proposition A.1:

Proposition A.4. *The domain of the outer contact billiard map over the loxodromic torus $\Sigma_c = \{c(xy+uv) = yu - xv\} \subset \mathbb{RP}^3$ is the region foliated by the quadrics:*

$$I_{\sigma^2} = \left\{ (1+c^2)\sigma^2 = \frac{c(xy+uv)+xv-yu}{c(xy+uv)-xv+yu} \right\} \subset \mathbb{RP}^3$$

for $\sigma \in (0,\infty)$. Each of these quadratic surfaces is invariant under the outer contact billiard map over Σ_c. On each leaf, the map is given by:

$$I_{\sigma^2} \ni (x:y:u:v) \mapsto (x':y':u':v') \in I_{\sigma^2}$$

by

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi & 0 & -\rho\sin\varphi & 0 \\ 0 & \frac{\cos\varphi}{\rho} & 0 & -\frac{\sin\varphi}{\rho} \\ \rho\sin\varphi & 0 & \rho\cos\varphi & 0 \\ 0 & \frac{\sin\varphi}{\rho} & 0 & \frac{\cos\varphi}{\rho} \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix},$$

where ρ, φ are constant on I_{σ^2}, determined through:

$$\rho = \sqrt{1 + \frac{4\sigma c}{\sigma^2 + (1-\sigma c)^2}}, \quad (\cos\varphi, \sin\varphi) = \frac{(1-(1+c^2)\sigma^2, 2\sigma)}{\sqrt{(1-(1+c^2)\sigma^2)^2 + 4\sigma^2}},$$

and there are no periodic orbits.

Remark 21. *The map is more compact in the complex variables $x+iu, y+iv$ where it is given by the (real) projectivization of*

$$(x+iu, y+iv) \mapsto (\rho e^{i\varphi}(x+iu), \rho^{-1}e^{i\varphi}(y+iv)).$$

Remark 22. *Similarly to the case of hyperbolic tori, although Proposition A.4 describes the domain as foliated by invariant loxodromic tori, these leaves are not disjoint. This time, any two leaves intersect along two disjoint projective lines*

$$(x:0:u:0) \ \ \textit{and} \ \ (0:y:0:v),$$

both contained in the table Σ_c. Since these lines are entirely inside the billiard table, removing Σ_c, Σ_c' leaves a genuine foliation of the billiard domain and the table has no singular points.

Lastly, item 6 from Proposition A.1:

Proposition A.5. *The domain of the outer contact billiard map over the parabolic torus $\Sigma = \{xy + uv = uy\} \subset \mathbb{RP}^3$ is the region foliated by the quadrics*

$$I_{\sigma^2} = \left\{ \sigma^2 = \frac{xy+uv+uy}{xy+uv-uy} \right\} \subset \mathbb{RP}^3$$

for $\sigma \in (0,\infty)$. Each of the quadratic surfaces is invariant under the outer contact billiard map over Σ. On each leaf, the map is given by

$$I_{\sigma^2} \ni (x:y:u:v) \mapsto (x':y':u':v') \in I_{\sigma^2}$$

as

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} \lambda & 0 & -\frac{1}{2}(\lambda^2-1) & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\lambda^2}) & 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix},$$

where $\lambda = \frac{1+\sigma}{1-\sigma}$*, is constant on* I_{σ^2}*. There are no periodic orbits.*

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