Álgebra lineal, parte 1: sistemas lineal

Its difficult to quickly impart the importance of linear algebra. In my experience this is because linear algebra is more of a basic tool which is *used* in other areas rather than a subject proper which is motivated purely by its own existence. More precisely:

- basically any mathematician you meet will answer 'yes' if you ask them if they use linear algebra in their research (this is natural: essentially everything is geometric in some form, and the most basic objects in geometry are points, lines, planes, ...etc. ie, linear objects).
- however, very few would say their research is purely in linear algebra (although research and open problems in pure linear algebra does certainly exist, e.g. in finding efficient algorithms for determinants/eigenvalues of large matrices).
- I think of the following analogy: linear algebra is like a nail, and we are interested in building some structures. Very few people dedicate their whole life to studying nails. Most of us are interested in learning how to use the various types of nails properly (some are for wood, some for cement, ...etc.). In this way, we may reliably put together and create a strong and intricate structure. At the end of the day, we are more interested in the types of structures we can assemble than how nails work...but if we never learn how to hammer a nail...it is slow to get started building.

For our start the relevant sections from our main references are:

- ch. 2 of Mota, Rumbos *Álgebra lineal en* $\mathbb{R}^n$,
- ch. 1 of Vargas, Wachtel *Álgebra lineal I.*

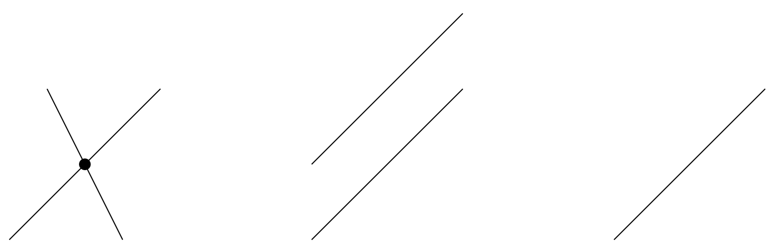
Contents

1 intersecting lines

We have seen the general equation for a line in the plane is (a *linear equation*)

$$ax + by = c$$

where a, b, c are some constants (and a, b are not both zero). The meaning of the equation is that the line consists of those pairs of numbers $(x, y) \in \mathbb{R}^2$ which satisfy the equation. Lines in the plane may intersect in:



a point (typical case), nowhere (parallels), a line! (repeated line),

To see the cases in this geometric picture with algebra (in equations) we consider two lines as given by a pair of linear equations:

$$ax + by = c$$

$$a'x + b'y = c'$$

where a, b, c and a', b', c' are given constants, and we are trying to find pairs of numbers x, y satisfying both equations. Lets suppose we had some values x, y satisfying this pair of equations. Then we multiply the 1st equation by a' and the 2nd by a to have:

$$aa'x + ba'y = ca',$$

$$aa'x + ab'y = ac'.$$

If we subtract these two equations we have (elimination of x's)

$$(ab' - a'b)y = ac' - a'c.$$

The same reasoning (multiplying 1st by b' and 2nd by b to then subtract and eliminate the y's) yields:

$$(ab' - a'b)x = b'c - bc'$$

Comparing these last two equations we see that the quantity:

$$\Delta = \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b$$

(the *determinant* of the system) plays a key role in which type of solution we may have. Namely:

- the typical case is when $\Delta \neq 0$. Then the pair of lines have a unique intersection point at

$$x = \frac{\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}}{\Delta} = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}{\Delta} = \frac{ac' - a'c}{ab' - a'b}.$$

- the exceptional cases can occur when $\Delta = 0$. Then we may have either an empty set (parallel lines) or a whole line (repeated line). The lines are parallel when $\Delta = 0$, and $ac' \neq a'c$, or $b'c \neq bc'$. The lines are the same when $\Delta = 0$ and also $ac' = a'c$ and $b'c = bc'$. Exercise: convince yourself!

Remark. *For a slightly longer exercise: carry out the analogous steps in 3-dimensional space. Namely, to get the ball rolling, we can start from the general equation for a plane (in space) $\mathbb{R}^3 \ni (x, y, z)$ is by an equation of the form*

$$ax + by + cz = d.$$

for a, b, c, d some constants. Three planes in space may intersect in

- *a point (typical case),*
- *a line,*
- *a plane (when the 'three' planes are actually the same: repeated),*
- *an empty set.*

To find these cases algebraically, consider three planes given by a triple of equations:

$$ax + by + cz = d$$
$$a'x + b'y + c'z = d'$$
$$a''x + b''y + c''z = d''$$

and determine for example conditions on these coefficients $(a, b, c, d, a', b', c', d', a'', b'', c'', d'')$ so that the three planes intersect in a unique point.

2 linear systems: definitions, notation

Generlizing the situation from the last section:

Definition 1. *A* linear system *of m-equations in n-variables is a list of equations of the form:*

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2,$$
$$...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m,$$

where the coefficients a_{jk}, b_j are some constants. A solution of the system is an n-tuple (list) of numbers $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ satisfying the m-equations simultaneously. The system is often written more compactly in matrix form which consists of the notation:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

and the system is summarized compactly with the notation: $Ax = b$.

For the equations in matrix form, the grid, A, of numbers with m rows and n columns (whose entries are the coefficients a_{jk} of the x-variables in the system) is called an $m \times n$ (read: m by n) *matrix*.

The system $Ax = b$ is also often represented by an *augmented matrix*, which holds the x-coefficients in A and the prescribed constant coefficient b values by placing them together as $[A \mid b]$, or, more explicitly:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{array} \right).$$

For instance, our first example of a pair of lines in the plane: $ax + by = c$, $a'x + b'y = c'$ (two equations in two unknowns) can be written in these notations with a 2×2 coefficient matrix as:

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ c' \end{pmatrix}$$

or, placing the relevant coefficients together in the 2×3 augmented matrix:

$$\left(\begin{array}{cc|c} a & b & c \\ a' & b' & c' \end{array} \right).$$

3 elimination (Gauss), echelon forms.

When we analyzed a pair of lines in the plane, we used some standard operations to 'eliminate' or simplify the equations. These basic steps, also called *elementary operations* consist in:

1. change the order in which the equations are written,
2. multiply an equation by a (non-zero) scalar (real number),
3. add or subtract two equations from eachother.

The main point being that these steps do not effect the solutions. Thus, after applying these steps to simplify the equations, we may more easily describe the solutions (or abscence of solutions) by simple inspection.

Remark. *The first two operations can be combined by saying: we may add to an equation any multiple of some other equation.*

After some examples, it should be convincing that by applying the above operations, any system of linear equations can be reduced to one of the following 'normal forms':

Definition 2. *A matrix is in row echelon form (*forma escalonada*:* **FE***) when:*

1. *If a row of the matrix contains only zeroes, then any row below it also contains only zeroes.*
2. *If a row has a non-zero entry then its first entry (from left to right) is normalized to be equal to one. Such an entry is called a pivot (*principal*).*
3. *The pivot in a given row is located below and to the right of any pivots in the preceding rows.*

*The matrix is said to be in reduced row echelon form (*forma escalonada reducida*:* **FER***) when it is in* **FE** *form and as well:*

4. *Each pivot is the only non-zero entry in its column.*

The definition is most easy to understand by seeing some examples:

- the matrix $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is in **FE** form. It is not in **FER** form since there is a 2 above the pivot in the bottom row.
- the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ is not in **FE** form: the pivot in the bottom row is not located to the right of the pivot in the preceding row.
- the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ is not in **FE** form: the first row of zeroes is not at the bottom, applying a row swap to $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ we are **FE** (and **FER**).
- the matrix $\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not in **FE** form: in the 2nd row, the first non-zero entry is 2 (and not one). Dividing the 2nd row by two (multiplying by $\frac{1}{2}$) we have $\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is in **FE** (and **FER**) form.

Definition 3. *The process of applying the elementary operations to convert a given matrix to* **FE** *form is called* Gaussian elimination*. Continuing to* **FER** *form is called* Gauss-Jordan elimination*. The number of pivots in* **FE** *(or* **FER***) form is called the* rank *of the matrix.*

Given an initial matrix A, its reduction to **FER** form is unique (see thm. 2.28 in Mota/Rumbos).

4 a few examples

- for the system of 3 equations in two unknowns:

$$2x + y = 3, \;\; x + y = 2, \;\; 2x + 3y = 6$$

in matrix form, $A\mathbf{x} = b$, as:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

or augmented matrix form $[A \mid b]$ as:

$$\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 6 \end{array}\right) \sim_{R_1 \to R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 6 \end{array}\right) \sim_{R_2 \to R_2 - R_1, \; R_3 \to R_3 - 2R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 4 \end{array}\right)$$

$$\sim_{R_3 \to R_3 - 3R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

and so there are *no* solutions (the last row reads as the impossible equation $0 = 1$).

- for the system of 3 equations in two unknowns:

$$3x + y = 7, \;\; 2x + y = 5, \;\; x + 2y = 4$$

in matrix form as:

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}$$

or augmented matrix form $[A \mid b]$ as:

$$\left(\begin{array}{cc|c} 3 & 1 & 7 \\ 2 & 1 & 5 \\ 1 & 2 & 4 \end{array}\right) \sim_{R_1 \to R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 1 & 2 & 4 \end{array}\right) \sim_{R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}\right)$$

$$\sim_{R_3 \to R_3 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

and we have the unique solution $x = 2, y = 1$.

- for the system of 3 equations in 4 unknowns:

$$3x+3y-3w=3,\ 2x+2y+z-w=3,\ x+y+z=2$$

in matrix form as

$$\begin{pmatrix} 3 & 3 & 0 & -3 \\ 2 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$$

or augmented matrix form $[A \mid b]$ as:

$$\left(\begin{array}{cccc|c} 3 & 3 & 0 & -3 & 3 \\ 2 & 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & 0 & 2 \end{array}\right) \sim_{R_1 \to \frac{1}{3}R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 2 & 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & 0 & 2 \end{array}\right)$$

$$\sim_{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \sim_{R_3 \to R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so we have a 2-dimensional solution space (a plane), by:

$$x = 1 - y + w,\ z = 1 - w$$

where x, y are dependent variables (*variables basicas, o, dependientes*), and y, w are free (*variables libres*). We can also describe the solution space in set notation as:

$$\{x, y, z, w \in \mathbb{R} \mid x = 1 - y + w,\ z = 1 - w\},$$

or, in vector form (take the free variables as parameters)

$$\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R}\}.$$

5 counting solutions, homogeneous equation

A general linear system (def. 1), $Ax = b$ with $b \neq 0$, may have *no solutions* (be inconsistent). A simpler type of systems are:

Definition 4. *A* homogeneous linear system *(of m-equations in n-variables) is one of the form:*

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0,$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0,$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0,$$

where the coefficients a_{jk} are some constants. A solution of the system is an n-tuple (list) of numbers $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ satisfying the m-equations simultaneously. The system is often written more compactly in matrix form which consists in taking:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and the system is summarized compactly with the notation: $Ax = 0$.

To describe the solutions to a homogeneous system, there is no point in writing the augmented matrix (we would just be carrying an extra column of zeroes along for the ride).

Homogeneous systems can *never* be inconsistent (they always admit, at least, the trivial solution $x_1 = 0, x_2 = 0, ..., x_n = 0$).

To count the dimension of the solution space to a homogeneous linear system we may reduce the $m \times n$ matrix A to **FE** (or **FER**) form and count the number of pivots (the rank), $r = \text{rank}(A) \leq n$. The space of solutions then has dimension equal to the number of remaining free variables: $n - r$.

Remark. *Recall that the rank of A, denoted rk(A), is the number of pivots in its reduction to* **FE** *(or* **FER***) form. The number of remaining free variables (which do not correspond to pivots) is also called the* nullity *of A (denoted nl(A)). So, for an $m \times n$ matrix A we always have:*

$$rk(A) + nl(A) = n.$$