Álgebra lineal, parte 2: espacios vectoriales[1]

We continue following the relevant sections from our main references:

- ch. 3, 4 of Mota, Rumbos *Álgebra lineal en* $\mathbb{R}^n$,
- ch. 3, 4 of Vargas, Wachtel *Álgebra lineal I.*

As opposed to the previous part, where we have made ourselves familiar with some basic computational aspects of manipulating systems of equations and matrices, here we will present more formally some basic underlying structure and general principles in *vector spaces*.

Contents

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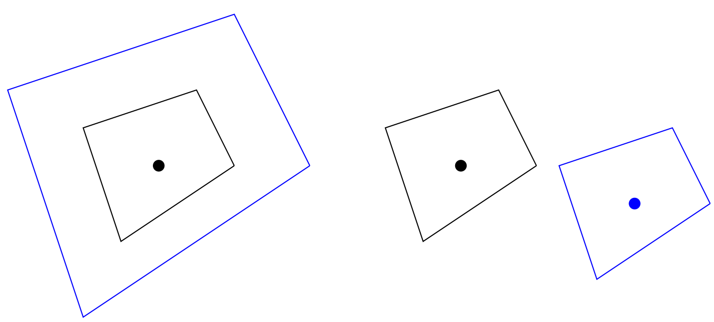
1 Vector spaces

The model example of a vector space is $\mathbb{R}^n$ (an n-dimensional real vector space). Recall: this is the set of all ordered lists of n real numbers $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ with each $x_j \in \mathbb{R}$ a real number. As is customary in linear algebra we will, unless mentioned otherwise, consider elements of $\mathbb{R}^n$ as *column vectors*, denoting them with either an over arrow or boldface to remind we have a vector:

$$\mathbf{x} = \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

From your courses in analytic geometry, the cases $n = 2$ and $n = 3$ are familiar (Cartesian coordinates for the Euclidean plane as $\mathbb{R}^2$, or the Euclidean space as $\mathbb{R}^3$). While each of these spaces have certain differences, they have enough shared features that it is worthwhile to 'axiomatize' these common features so that we have a common ground of properties which hold regardless of what particular n is under consideration.

Geometrically, these first properties consist in the operations of *scaling* and *translating*:



(a) Re-scaling a figure. (b) Translating a figure.

Algebraically, for given n and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, a \in \mathbb{R}$, we scale and translate by:

$$a\mathbf{x} := \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{pmatrix}, \quad \mathbf{x} + \mathbf{y} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \tag{1}$$

And the essential properties of these operations are axiomatized in the following general definition of an 'abstract vector space':

Definition 1. *A set V is equipped with the structure of a real* vector space *when it has two well defined operations:*

$$V \times V \to V, \quad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}, \qquad \mathbb{R} \times V \to V, \quad (a, \mathbf{v}) \mapsto a\mathbf{v}$$

called vector addition and scalar mutliplication resp., which satisfy:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},\ \forall \mathbf{u}, \mathbf{v} \in V$,
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}),\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,
3. $\exists\ \mathbf{0} \in V$, *such that* $\mathbf{v} + \mathbf{0} = \mathbf{v},\ \forall \mathbf{v} \in V$,
4. $\forall \mathbf{v} \in V$, $\exists - \mathbf{v} \in V$ *such that* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. $1\mathbf{v} = \mathbf{v},\ \forall \mathbf{v} \in V$,
6. $(ab)\mathbf{v} = a(b\mathbf{v}),\ \forall a, b \in \mathbb{R},\ \mathbf{v} \in V$,
7. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v},\ \forall a, b \in \mathbb{R},\ \mathbf{v} \in V$,
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v},\ \forall a \in \mathbb{R},\ \mathbf{u}, \mathbf{v} \in V$

Examples.

- $\mathbb{R}^n$ with vector addition and scalar multiplication defined by (1) is a real vector space (exercise: check!)
- Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of functions from $\mathbb{R} \to \mathbb{R}$. Then, given $a \in \mathbb{R}$ and $f, g \in V$ (that is two functions $f, g : \mathbb{R} \to \mathbb{R}$) we can define $f + g, af \in V$ by:

  $$(f + g)(x) := f(x) + g(x), \quad (af)(x) := a(f(x)).$$

  With these operations, $\mathcal{F}(\mathbb{R}, \mathbb{R})$, is a real vector space (exercise: check!).
- Consider the same set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ from the previous example but with

  $$(f + g)(x) := f(g(x)), \quad (af)(x) := a(f(x)).$$

  With these operations we do not have a vector space structure (exercise: which of the 'axioms' in definition 1 above fail?).
- Consider the same set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ from the previous example but with

  $$(f + g)(x) := f(x)g(x), \quad (af)(x) := a(f(x)).$$

  With these operations we do not have a vector space structure (exercise: which of the 'axioms' in definition 1 above fail?).

As some 'comforting' consequences of the vector space axioms, for example:

Proposition 1. *In a vector space V, there is a unique element $\mathbf{0} \in V$ (called* the *zero vector) satisfying item 3 of definition 1*

Proof. Suppose $\mathbf{0}, \mathbf{0}' \in V$ satisfy item 3 of the definition. Then:

$$\mathbf{0} \overset{3}{=} \mathbf{0} + \mathbf{0}' \overset{1}{=} \mathbf{0}' + \mathbf{0} \overset{3}{=} \mathbf{0}'$$

so that in fact $\mathbf{0} = \mathbf{0}'$ and the zero vector is unique. $\square$

Proposition 2. *In a vector space V, for any $\mathbf{v} \in V$ then:*

$$0\mathbf{v} = \mathbf{0}.$$

Proof. Let $\mathbf{v} \in V$. Since we already know $0 \in \mathbb{R}$ has $0 + 0 = 0$ then:

$$0\mathbf{v} + 0\mathbf{v} \overset{7}{=} (0 + 0)\mathbf{v} = 0\mathbf{v}$$

By item 4 in the definition, there exists some $\mathbf{u} = -(0\mathbf{v}) \in V$ such that $0\mathbf{v} + \mathbf{u} = \mathbf{0}$. Now adding $\mathbf{u}$ to both sides of $0\mathbf{v} + 0\mathbf{v} = 0\mathbf{v}$ we have:

$$0\mathbf{v} \overset{3}{=} 0\mathbf{v} + \mathbf{0} = 0\mathbf{v} + ((0\mathbf{v}) + \mathbf{u}) \overset{2}{=} (0\mathbf{v} + 0\mathbf{v}) + \mathbf{u} = 0\mathbf{v} + \mathbf{u} = \mathbf{0}.$$

$\square$

EXERCISES.

Show, from the vector space axioms, the following properties:

- For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ such that $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
- For any $a \in \mathbb{R}$ then $a\mathbf{0} = \mathbf{0}$.
- For $\mathbf{v} \in V, a \in \mathbb{R}$, then $a\mathbf{v} = 0 \iff a = 0$ or $\mathbf{v} = \mathbf{0}$.
- For $\mathbf{v} \in V$, there is a unique $-\mathbf{v} \in V$ satisfying item 4 of definition 1.
- For $\mathbf{v} \in V$ then $(-1)\mathbf{v} = -\mathbf{v}$.
- $\mathbf{v} = -\mathbf{v} \iff \mathbf{v} = \mathbf{0}$.
- If a real vector space V has more than one element, then it has infinitely many elements.

Remark. *For $\mathbf{u}, \mathbf{v} \in V$, its common to use the more concise notation:*

$$\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v}).$$

Remark. *A real vector space structure allows us to form simple linear equations, such as:*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k = \mathbf{u}$$

in which we have been asking questions such as: given the vectors $\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{u}$, for which values of the coefficients $a_1, ..., a_k \in \mathbb{R}$ does the equation hold true?

Remark. *We defined above (def. 1) a* real *vector space structure on a set V. Likewise, one can replace the scalars $\mathbb{R}$ by some other number system to have analogous structures (for example the structure of a* complex *vector space).*

2 Subspaces

A given real vector space V may contain important subsets: distinguished by their compatibility with V's vector space structure. For example:

- a line (through the origin) in the Cartesian plane $\mathbb{R}^2$,
- a plane (through the origin) in the Cartesian space $\mathbb{R}^3$.

These spaces are interesting because they are themselves vector spaces, when equipped with the restriction of V's operations. Formally:

Definition 2. *A (non-empty) subset $U \subset V$ of a vector space V is called a* subspace *of V when V's operations of vector addition and scalar multiplication restricted to elements of U induce a real vector space structure on U.*

Examples.

- An affine line in $\mathbb{R}^2$ which does not pass through the origin is *not* a subspace of $\mathbb{R}^2$. The difference between considering (vector) subspaces vs. affine subspaces amounts algebraically to the difference between considering homogeneous systems of linear equations vs. general (not necessarly homogeneous) systems of linear equations. Geometrically, affine spaces (vector spaces without any 'distinguished point' as the origin) are more natural, but algebraically (formulas) it is simpler to consider vector spaces, where we have a distinguished origin and may work (most often) with the simpler homogeneous equations.
- The set of solutions to any homogeneous system of linear equations form a vector subspace (which may be the 'trivial' subspace: $\{\mathbf{0}\} \subset V$, there is another 'trivial' subspace of any vector space V...which is just $V \subset V$).
- Any subspace always contains, at least the zero vector.
- The union of the x and y axes in $\mathbb{R}^2$ is *not* a subspace (what about their intersection?).
- The union of the xz and yz planes in $\mathbb{R}^2$ is *not* a subspace. Their intersection (the z-axis) *is* a subspace.

As a convenient characterization of subspaces:

Proposition 3. *A (non-empty) subset $U \subset V$ is a subspace of V iff U is closed under V's vector addition and scalar multiplication. That is, when for any $\mathbf{u}, \mathbf{u}' \in U$ and $a \in \mathbb{R}$ we have:*

1. $\mathbf{u} + \mathbf{u}' \in U$,
2. $a\mathbf{u} \in U$.

Proof. ($\Longrightarrow$) If U is a subspace of V then, in order to inherit well-defined addition and scaling operations from the restriction of V's, we must have, at least, that U be closed under addition and scaling.

($\Longleftarrow$) Conversely, when U is closed under addition and scaling, these restrictions satisfy items 1,2,5,6,7,8 of a real vector space structure (def. 1) since those of V do (exercise). To see 3 and 4 we can note that for $\mathbf{u} \in U$ then $0\mathbf{u} = \mathbf{0} \in U$ since U is closed under scaling. Still this element has $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in U$ so that 3 holds. Likewise $(-1)\mathbf{u} = -\mathbf{u} \in U$ since U is closed under scaling, and still $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ so that 4 holds. $\square$

Exercises.

- Is $\mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\} \subset \mathbb{R}^2$ a subspace? Which items in def. 1 fail?
- Let $U_1, U_2 \subset V$ be two subspaces of a vector space V.
  - Show their intersection $U_1 \cap U_2$ is always a subspace.
  - Is their union $U_1 \cup U_2$ always a subspace? Sometimes a subspace?
- Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of functions from $\mathbb{R} \to \mathbb{R}$ with the vector space structure defined above (of pointwise addition and scaling).
  - Is $\mathcal{P}_3 = \{x \mapsto p(x)$ s.t. $p(x)$ is some polynomial of degree at most $3\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
  - Is $\mathcal{P}'_3 = \{x \mapsto p(x)$ s.t. $p(x)$ is some polynomial of degree $3\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
  - Is $\mathcal{F}_0 = \{x \mapsto f(x)$ s.t. $f(0) = 0\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
  - Is $U = \{x \mapsto f(x)$ s.t. $f(0) = 2f(1)\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
  - Is $\mathcal{F}_+ = \{x \mapsto f(x)$ s.t. $f(x) \geq 0\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
  - Is $Per = \{x \mapsto f(x)$ s.t. $f(x+1) = f(x)\}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?
- Show the only subspaces of $\mathbb{R}^3$ are those from the following list:

  (a) the 'trivial' subspace $\{\mathbf{0}\}$,

  (b) a line through the origin,

  (c) a plane through the origin,

  (d) the 'trivial' subspace $\mathbb{R}^3$.

3 generating subspaces (span)

A common way to define a subspace is by giving some system of vectors 'generating' the subspace:

- To specify a line through the origin of $\mathbb{R}^2$, we can give some generating vector, $\mathbf{v}$, directing said line: all points in the line are multiples of said generator ($\{t\mathbf{v} : t \in \mathbb{R}\}$).

  For example the line $\{y = x\} \subset \mathbb{R}^2$ is generated by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- To specify a plane through the origin in $\mathbb{R}^3$, we can give a pair of generating vectors, $\mathbf{u}, \mathbf{v}$, in said plane: all points in the plane are some sum of multiples of said vectors ($\{s\mathbf{u} + t\mathbf{v} : s, t \in \mathbb{R}\}$).

  For example the xy plane $\{z = 0\} \subset \mathbb{R}^3$ is generated by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

- Question: are generator(s) of a given line(plane) unique?

- Comment: specifying a subspace with some generators is not the *only* option. For instance we might specify a plane in $\mathbb{R}^3$ as some level set, or with some normal vector.

More formally:

Definition 3. *Let $X \subset V$ be some subset of a vector space V. The subspace, $span\{X\}$, of V* generated *by X (or* spanned *by the elements of X) is the smallest subspace of V containing X:*

$$span\{X\} := \bigcap_{\{U : U \supset X,\ and\ U\ is\ a\ subset\ of\ V\}} U$$

Remark. *Any intersection of subspaces is a subspace (exercise: prove it!) so that $span\{X\}$ as defined above is always a subspace (with $V \supset span\{X\} \supset X$).*

Examples.
For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then for $X = \{\mathbf{u}, \mathbf{v}\}$ we may have:

- if $\mathbf{u} = \mathbf{v} = 0$ then $\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{0}\} = \{\mathbf{0}\}$.

- if $\mathbf{u} \neq 0$, but $\mathbf{v} = a\mathbf{u}$ is a multiple of $\mathbf{u}$, then $\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u}\}$ is the line through the origin directed by $\mathbf{u}$.

- if $\mathbf{u} \neq 0$, $\mathbf{v} \neq 0$ are not multiples of eachother, then $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is the plane through the origin directed by $\mathbf{u}, \mathbf{v}$.

We will primarily be interested in subspaces generated by a finite number of vectors. Calling:

Definition 4. *Given some vectors* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in V$ *in a vector space* V *a* linear combination *of said vectors is a vector of the form:*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k$$

for some scalars $a_1, a_2, ..., a_k \in \mathbb{R}$.

Then subspaces generated by a finite number of vectors are, explicitly:

Proposition 4. *For a vector space* V *and given vectors* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in V$ *then:*

$$span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k : a_1, a_2, ..., a_k \in \mathbb{R}\}$$

is the subspace consisting of all possible linear combinations of said vectors.

Proof. We must show that $U_* = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k : a_1, a_2, ..., a_k \in \mathbb{R}\}$ is (1) a subspace of V containing the $\mathbf{v}_j$ and (2) any subspace U of V containing $\mathbf{v}_j \in U$ contains U_* (ie $U \supset U_*$).

For (1), by taking $a_j = 1$ and all others zero we see that $\mathbf{v}_j \in U_*$. To see U_* is a subspace we check that U_* is closed under addition and scaling. Indeed, let $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k \in U_*$ be some vector in U_* and $a \in \mathbb{R}$ some scalar. Then:[2]

$$a(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k) = aa_1\mathbf{v}_1 + aa_2\mathbf{v}_2 + ... + aa_k\mathbf{v}_k \in U_*$$

since $\tilde{a}_j = aa_j \in \mathbb{R}$. Likewise if $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + ... + b_k\mathbf{v}_k \in U_*$ is some other vector in U_* then we have

$$a_1\mathbf{v}_1 + ... + a_k\mathbf{v}_k + b_1\mathbf{v}_1 + ... + b_k\mathbf{v}_k = (a_1 + b_1)\mathbf{v}_1 + ... + (a_k + b_k)\mathbf{v}_k \in U_*$$

since $a'_j = a_j + b_j \in \mathbb{R}$. So indeed (prop. 3) U_* is a subspace containing the $\mathbf{v}_j$'s.

For (2), if $U \subset V$ is some subspace containing all $\mathbf{v}_j \in U$, then since U is closed under scaling and addition we must have also $a_j\mathbf{v}_j \in U$ (for any $a_j \in \mathbb{R}$) and as well any sum $a_1\mathbf{v}_1 + ... + a_k\mathbf{v}_k \in U$. That is, we must have all the general elements of U_* in U, ie $U_* \subset U$. □

3.1 linear systems revisited

We have studied before linear systems in the following explicit form:

$$b_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n,$$

$$b_2 = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n,$$

$$...$$

$$b_m = a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n$$

[2]Here we'll use the distributive/associative laws for k elements, which follow from the vector space axioms and induction.

(a system of m equations in n variables). We can write the system more compactly in vector/matrix notation as:

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = A\mathbf{x} \tag{2}$$

where

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \ \mathbf{a}_j = \begin{pmatrix} a_{j1} \\ a_{j1} \\ \vdots \\ a_{jm} \end{pmatrix} \in \mathbb{R}^m, \ \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

and A is the $m \times n$ matrix with columns $\mathbf{a}_j,\ j = 1, ..., n$.

This linear system (2) makes sense in any vector space and we can re-phrase it with our new language in the following ways:

- given $n+1$ vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n, \mathbf{b} \in V$, what are the scalars $x_1, x_2, ..., x_n \in \mathbb{R}$ so that (2) holds?

- given $n+1$ vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n, \mathbf{b} \in V$, the consistency of the system amounts to asking: when is

  $$\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}?$$

- given $n+1$ vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n, \mathbf{b} \in V$, consider the (linear) map:

  $$A : \mathbb{R}^n \to V, \ \ \mathbf{x} \mapsto x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = A\mathbf{x}.$$

  The consistency of the system amounts to asking: when is $\mathbf{b} \in \text{im}(A)$? Recall the *image* of a map A is $\text{im}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$. The solution space is the *pre-image* $A^{-1}\{\mathbf{b}\} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$.

3.2 some examples

- Here are three vectors in $\mathbb{R}^3$:

  $$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

  Do these vectors generate all of $\mathbb{R}^3$?

  This question is the same as one we have already considered: whether a certain linear system is always consistent. In particular we can settle it by reducing an appropriate matrix to echelon form.

  Namely, we are asking whether for any $\mathbf{b} \in \mathbb{R}^3$ there always exist some $x_1, x_2, x_3 \in \mathbb{R}$ so that:

  $$\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = A\mathbf{x}$$

for A the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So we form the augmented matrix and reduce (exercise: perform the Gauss-Jordan elimination!):

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & b_1 \\ 1 & 2 & 2 & b_2 \\ -1 & 2 & 2 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(b_2 - b_3) \\ 0 & 1 & 0 & -b_1 + \frac{1}{2}(3b_2 - b_3) \\ 0 & 0 & 1 & b_1 - \frac{1}{4}(5b_2 - 3b_3) \end{array}\right)$$

Since the matrix has full rank, it has no inconsistencies, and thus there are always solutions (for any $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ we have $x_1 = \frac{1}{2}(b_2 - b_3)$, $x_2 = -b_1 + \frac{1}{2}(3b_2 - b_3)$, $x_3 = b_1 - \frac{1}{4}(5b_2 - 3b_3)$, or to summarize:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3.$$

- Same, for the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

we ask: do these vectors generate all of $\mathbb{R}^3$?

To settle it, we form the augmented matrix and reduce:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ -1 & 2 & 3 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{3}(2b_1 - b_2) \\ 0 & 1 & 1 & \frac{1}{3}(2b_2 - b_1) \\ 0 & 0 & 0 & 4b_1 - 5b_2 + 3b_3) \end{array}\right)$$

the system has sub-maximal rank (two) and thus there are possible inconsistencies. More precisely whenever $4b_1 - 5b_2 + 3b_3 \neq 0$ there are no solutions, ie such $\mathbf{b}$ are *not* generated by these three vectors and these three vectors *do not generate* $\mathbb{R}^3$. In fact the only vectors that are generated by this triple are precisely those lying in the plane thru the origin: $4b_1 - 5b_2 + 3b_3 = 0$, or summarizing:

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3 : 4b_1 - 5b_2 + 3b_3 = 0\} \subset \mathbb{R}^3.$$

4 bases (dependence/independence), dimension

There are many sets of vectors which can generate the same subspace, for instance:

- $\operatorname{span}\{\mathbf{v}\} = \operatorname{span}\{\mathbf{v}, 2\mathbf{v}\} = \operatorname{span}\{\mathbf{v}, 2\mathbf{v}, 3\mathbf{v}\} = \operatorname{span}\{2\mathbf{v}\}$
- $\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \operatorname{span}\{\mathbf{u}, 2\mathbf{v}\} = \operatorname{span}\{\mathbf{u}, 2\mathbf{v}, 3\mathbf{v}\} = \operatorname{span}\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{v}\}$

Given some subspace, it is natural to ask for a set of generating vectors which is 'most efficient' in the sense that the length of the list of generating vectors is minimal. The length of such a list is, when finite, unique and captures the dimension of the subspace.

In order to minimize the length of a list of generating vectors, we may produce an algorithm based on the following observation: for a set of vectors generating a given subspace, in order to potentially shorten the set of generators, we note that if any vector in the list is a linear combination of the remaining vectors in the list, then we may throw it out without effecting the span:

Proposition 5. *span*$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} =$ *span*$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{u}\}$ *whenever*

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k \in \textit{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}.$$

Proof. Its clear that we always have $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \subset \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{u}\}$ since any linear combination of the $\mathbf{v}_j$'s is also a linear combination of the $\mathbf{v}_j$'s and $\mathbf{u}$ (put the coefficient of $\mathbf{u}$ as zero). Conversely, supposing that $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ then a general element $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k + a\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{u}\}$, by re-arranging we have $\mathbf{w} = (a_1 + ac_1)\mathbf{v}_1 + ... + (a_k + ac_k)\mathbf{v}_k \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ so that also $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} \supset \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k, \mathbf{u}\}$. $\square$

Note that the condition $\mathbf{u} \in \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ can be written more symmetrically as:

$$\mathbf{0} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k - \mathbf{u}$$

and a more democratic definition (without given preference to any 'distinguished' vector in the list) is formalized by:

Definition 5. *A list of vectors* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ *are called* dependent *if there exists scalars* $c_1, c_2, ..., c_k \in \mathbb{R}$ *not all zero such that*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}.$$

Definition 6. *Likewise we call a list of vectors* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ independent *when*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0} \implies c_1 = c_2 = ... = c_k = 0.$$

Remark. *To test whether a given list of vectors (in* $\mathbb{R}^n$ *say) are dependent, we form the matrix,* A*, with columns* $\mathbf{v}_1, ..., \mathbf{v}_k$ *and ask whether the linear equation* $A\mathbf{x} = \mathbf{0}$ *has any non-zero solutions, in other words, after reducing* A *to* **FER** *form whether there are any free variables.*

Remark. *If* $\mathbf{v}_1, ..., \mathbf{v}_k$ *are dependent then, by definition, there are some scalars* $c_1, ..., c_k \in \mathbb{R}$ *not all zero so that* $\mathbf{0} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$*. If say* $c_j \neq 0$ *then this last equation can be re-arranged as:*

$$\mathbf{v}_j = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_{j-1}\mathbf{v}_{j-1} + a_{j+1}\mathbf{v}_{j+1} + ... + a_k\mathbf{v}_k$$

where $a_i = -c_i/c_j$*. In other words* $\mathbf{v}_j \in$ *span*$\{\mathbf{v}_1, ..., \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, ..., \mathbf{v}_k\}$ *and so in particular our list of generators can be shortened (by throwing out* $\mathbf{v}_j$*).*

So to test whether a given list of vectors generating a subspace can be shortened we ask whether there exist dependent vectors in the list (in case there are, the list can be shortened by throwing at least one vector out and then with the shortened list, we repeat and ask the question again). The length of an initial finite list will then always be shortened until the subspace is spanned by some list of independent vectors, and the algorithm terminates. The following proposition says that the length of this list of independent vectors is minimal:

Proposition 6. *If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_d$ are independent, then*

$$span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_d\} = span\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_l\}$$

implies $l \geq d$.

Proof. Suppose that $l < d$. We 'replace from the left' to obtain a contradiction. Set $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_d\}$ and consider the list $\mathbf{v}_1, \mathbf{u}_1, ..., \mathbf{u}_l$. This list still spans U. However it is not an independent list since the $\mathbf{u}$'s already span U, so e.g. $\mathbf{v}_1 = a_1\mathbf{u}_1 + ... + a_l\mathbf{u}_l$. Moreover, these $a_1, ..., a_l$ cannot all be zero since by assumption the $\mathbf{v}$'s are independent (so, at least, $\mathbf{v}_1 \neq 0$). Thus in particular we can solve for some $\mathbf{u}_{j_1}$ to be in the span of the remaining $\mathbf{u}$'s and $\mathbf{v}_1$, and remove this $\mathbf{u}_{j_1}$ without effecting the span to have U spanned by $\mathbf{v}_1$ and $\{\mathbf{u}_1, ..., \mathbf{u}_l\} \backslash \{\mathbf{u}_{j_1}\}$. Now we repeat and consider the list with $\mathbf{v}_1, \mathbf{v}_2$ and $\{\mathbf{u}_1, ..., \mathbf{u}_l\} \backslash \{\mathbf{u}_{j_1}\}$ which still spans U. Again, this list cannot be independent, since e.g. $\mathbf{v}_2 = c_1\mathbf{v}_1 + a_1\mathbf{u}_1 + ... + a_{j_1-1}\mathbf{u}_{j_1-1} + a_{j_1+1}\mathbf{u}_{j_1+1} + ... + a_l\mathbf{u}_l$. Again the a_j's cannot all vanish, since the $\mathbf{v}$'s are independent, so that some $\mathbf{u}_{j_2}$ is a linear combination of the $\mathbf{v}_1, \mathbf{v}_2$ and the remaining $\mathbf{u}$'s. Removing this $\mathbf{u}_{j_2}$ we have a spanning set of U with $\mathbf{v}_1, \mathbf{v}_2$ and $\{\mathbf{u}_1, ..., \mathbf{u}_l\} \backslash \{\mathbf{u}_{j_1}, \mathbf{u}_{j_2}\}$. Since we have assumed, for sake of contradiction, that $l < d$, we repeat this process of replacing $\mathbf{u}$'s by $\mathbf{v}$'s l-times until we have $U = \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_l\}$ where $l < d$. But then $U \ni \mathbf{v}_{l+1} = c_1\mathbf{v}_1 + ... + c_l\mathbf{v}_l$ contradicting independence of the $\mathbf{v}$'s. □

Due to the last property, we can define:

Definition 7. *Let V be a vector space which is generated by some finite number of vectors, $V = span\{\mathbf{v}_1, ..., \mathbf{v}_k\}$. The* dimension *of V is the minimal length of a generating list of vectors:*

$$\dim V := \min\{k : span\{\mathbf{v}_1, ..., \mathbf{v}_k\} = V\}.$$

A basis *of V is a list of generating vectors of this minimal length.*

Remark. *That is, if $\dim V = d$, then $\mathbf{v}_1, ..., \mathbf{v}_d$ is a basis for V when $V = span\{\mathbf{v}_1, ..., \mathbf{v}_d\}$. Such a list is necessarily independent so that equivalently a basis for V is a list of* independent *vectors generating (spanning) V.*

LOOSE ENDS: REMAINING EXAMPLES/EXERCISES.

- The *standard basis* of $\mathbb{R}^n$ is $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ were $\mathbf{e}_j$ is the vector with all zeroes except for a 1 in the j'th component.

- If $\mathbf{v}_1, ..., \mathbf{v}_d$ is a basis for V then any $\mathbf{v} \in V$ has a *unique* representation as $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_d\mathbf{v}_d$ in this basis (the *coordinates* associated to said basis).
- $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \in \mathbb{R}^2$ are dependent iff $ab' = a'b$.
- Let $U, U' \subset V$ be two subspaces of the vector space V.
  - If $U \subset U'$ and $\dim U = \dim U'$ then $U = U'$.
  - Show $\mathrm{span}\{U \cup U'\} = \{\mathbf{u} + \mathbf{u}' : \mathbf{u} \in U, \mathbf{u}' \in U'\}$. This subspace is commonly denoted $U + U' := \mathrm{span}\{U \cup U'\}$.
  - Show $\dim U + \dim U' = \dim(U + U') + \dim(U \cap U')$.
  - We call U, U' *complementary subspaces* when $U+U' = V$ and $U \cap U' = \{\mathbf{0}\}$ (common notation is $U \oplus U' = V$). Show that if U, U' are complementary then $\dim U + \dim U' = \dim V$.

4.1 some examples

- Here are three vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 2 \end{pmatrix}.$$

  Are they independent? dependent?

  Lets first note that (without doing any computations) we already know these vectors *cannot* generate $\mathbb{R}^4$ (or be a basis for $\mathbb{R}^4$) since there are only three of them (at most they generate a 3-dimensional subspace).

  To examine if these vectors are independent or dependent we are asking if there exists some $x_1, x_2, x_2 \in \mathbb{R}$ (not all zero) such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, that is, in matrix notation, if the homogeneous system:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

  has only the trivial solution ($x_1 = x_2 = x_3 = 0$, ie independent) or has some non-trivial solutions (dependent). This question we can answer by reducing the matrix to **FER** form:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and, having full rank, there are no free variables which is to say that there is a unique solution, namely, the trivial solution is the only one and *these three vectors are independent.*

- Here are four vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 4 \end{pmatrix}$$

  Are they independent? dependent? Do they span $\mathbb{R}^4$? Are they a basis for $\mathbb{R}^4$?

  We can first note that many of these questions are related, for example, if 4 vectors are span $\mathbb{R}^4$ then they are automatically independent and consequently a basis, or similarly 4 independent vectors in $\mathbb{R}^4$ are always a basis.

  So we will check whether these 4 vectors are independent (or dependent). As before this is equivalent to asking whether the homogeneous system $x_1\mathbf{v}_1 + ... + x_4\mathbf{v}_4 = \mathbf{0}$ has only the trivial solution (or has some free variables and non-trivial solutions). So we reduce:

$$\begin{pmatrix} 2 & 1 & -1 & 5 \\ 1 & 2 & 1 & 4 \\ -1 & 2 & 3 & 0 \\ 1 & 2 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  and we have free variables, so these vectors are dependent (consequently, they cannot span $\mathbb{R}^4$, nor can they be a basis...in fact the number of pivots –rank— here tells us there are exactly two independent vectors here so that these vectors generate a 2-dimensional plane inside $\mathbb{R}^4$).