

# ÁLGEBRA LINEAL: TRANSPUESTAS<sup>1</sup>

In this note we give some basic properties of the *transpose* of a matrix:

**Definition 1.** For  $A$  an  $m \times n$  matrix, its transpose is the  $n \times m$  matrix,  $A^T$ , where the  $j$ 'th row of  $A^T$  is the  $j$ 'th column of  $A$  (for  $j = 1, \dots, n$ ).

EXAMPLE.

$$\bullet \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

**Proposition 1.** Equivalently the  $k$ 'th column of  $A^T$ , is the  $k$ 'th row of  $A$  (for  $k = 1, \dots, m$ ). Consequently  $(A^T)^T = A$ .

*Proof.* Let  $a_{\mu,\nu}$  be the entry in  $A$ 's  $\mu$ 'th row and  $\nu$ 'th column ( $\mu \in \{1, \dots, m\}$  and  $\nu \in \{1, \dots, n\}$ ).

If we construct the transpose by the first definition above, where the  $\nu$ 'th row of  $A^T$  is the  $\nu$ 'th column of  $A$  then this  $\nu$ 'th row of  $A^T$  has entries  $\alpha_{\nu,\mu} = a_{\mu,\nu}$  as  $\mu \in \{1, \dots, m\}$ , so that the entry in  $A^T$ 's  $\nu$ 'th row and  $\mu$ 'th column is  $\alpha_{\nu,\mu} = a_{\mu,\nu}$ .

With the alternate, where the  $\mu$ 'th column of  $A^T$  is the  $\mu$ 'th row of  $A$  then this  $\mu$ 'th column of  $A^T$  has entries  $\alpha_{\nu,\mu} = a_{\mu,\nu}$  as  $\nu \in \{1, \dots, n\}$ , so that the entry in  $A^T$ 's  $\nu$ 'th row and  $\mu$ 'th column is again  $\alpha_{\nu,\mu} = a_{\mu,\nu}$ .  $\square$

We will show:

**Theorem 1.** For  $A, B$  some  $m \times n$  matrices, and  $C$  an  $l \times m$  matrix, then:

1.  $(aA + bB)^T = aA^T + bB^T$  (for any scalars  $a, b \in \mathbb{R}$ )
2.  $(CA)^T = A^T C^T$
3. if  $A$  is invertible, then so too is  $A^T$ , and  $(A^T)^{-1} = (A^{-1})^T$ .

**Theorem 2.** For  $A$  an  $m \times n$  matrix, with transpose  $A^T$  then:

1.  $\ker A = (\text{im}(A^T))^\perp$ ,
2.  $\ker A^T = \text{im}(A)^\perp$ .

Here for a subspace  $U \subset \mathbb{R}^k$  we write  $U^\perp = \{\mathbf{v} \in \mathbb{R}^k : \mathbf{v} \cdot \mathbf{u} = 0, \forall \mathbf{u} \in U\}$  for the orthogonal complement of  $U$  (for example the orthogonal complement of a plane in  $\mathbb{R}^3$  is its normal line).

Since we have seen the transpose is connected to the usual (Euclidean) inner product (for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  as column vectors, then  $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ ) we might expect there is some relation of transpose matrices and Euclidean operations such as orthogonal complements (as in theorem 2). Indeed, a unified way to establish properties of the transpose is using its characterizing relation to the inner product:

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**Proposition 2.** Let  $A$  be an  $m \times n$  matrix, with corresponding linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m, \mathbf{x} \mapsto A\mathbf{x}$ . Then the transpose matrix corresponds to the unique linear map  $\mathbb{R}^m \rightarrow \mathbb{R}^n, \mathbf{y} \mapsto A^T\mathbf{y}$  such that:

$$(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A^T\mathbf{y}), \quad \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m. \quad (1)$$

**Remark.** Note that the formula (1) may involve two different inner products. On the left,  $(A\mathbf{x}) \cdot \mathbf{y}$ , we have the inner product between two vectors in  $\mathbb{R}^m$  while on the right,  $\mathbf{x} \cdot (A^T\mathbf{y})$ , the inner product between two vectors in  $\mathbb{R}^n$ .

*Proof.* Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the standard basis of  $\mathbb{R}^n$  and  $\mathbf{E}_1, \dots, \mathbf{E}_m$  the standard basis of  $\mathbb{R}^m$ . The  $n$  columns of  $A$  are  $A\mathbf{e}_1, \dots, A\mathbf{e}_n$ , with entries:

$$A\mathbf{e}_\nu = \begin{pmatrix} a_{1,\nu} \\ a_{2,\nu} \\ \vdots \\ a_{m,\nu} \end{pmatrix}, \quad (\nu = 1, \dots, n)$$

and so  $a_{\mu,\nu} = (A\mathbf{e}_\nu) \cdot \mathbf{E}_\mu$  is the entry of  $A$  in column  $\nu$  and row  $\mu$ . In the same way,  $A^T\mathbf{E}_\mu$  is the  $\mu$ 'th column of  $A^T$ , and so  $\mathbf{e}_\nu \cdot (A^T\mathbf{E}_\mu) = a_{\nu,\mu}$  is the entry in the  $\nu$ 'th row and  $\mu$ 'th column of  $A^T$ , which (as in the proof of prop. 1) is exactly  $a_{\mu,\nu}$  which is to say we have  $(A\mathbf{e}_\nu) \cdot \mathbf{E}_\mu = \mathbf{e}_\nu \cdot (A^T\mathbf{E}_\mu)$  for any elements of these bases. Now, for general vectors it follows by 'bi-linearity', in stages:

- for any  $\mathbf{y} = y_1\mathbf{E}_1 + \dots + y_m\mathbf{E}_m \in \mathbb{R}^m$  then:

$$\begin{aligned} (A\mathbf{e}_\nu) \cdot \mathbf{y} &= y_1 A\mathbf{e}_\nu \cdot \mathbf{E}_1 + \dots + y_m A\mathbf{e}_\nu \cdot \mathbf{E}_m = \\ &= y_1 \mathbf{e}_\nu \cdot A^T\mathbf{E}_1 + \dots + y_m \mathbf{e}_\nu \cdot A^T\mathbf{E}_m = \mathbf{e}_\nu \cdot A^T\mathbf{y} \end{aligned}$$

so that, for any  $\mathbf{y} \in \mathbb{R}^m$  and  $\mathbf{e}_\nu$  in the basis we have  $(A\mathbf{e}_\nu) \cdot \mathbf{y} = \mathbf{e}_\nu \cdot (A^T\mathbf{y})$ .

- for any  $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$  then:

$$\begin{aligned} A\mathbf{x} \cdot \mathbf{y} &= x_1 A\mathbf{e}_1 \cdot \mathbf{y} + \dots + x_n A\mathbf{e}_n \cdot \mathbf{y} = \\ &= x_1 \mathbf{e}_1 \cdot A^T\mathbf{y} + \dots + x_n \mathbf{e}_n \cdot A^T\mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y} \end{aligned}$$

which is our claimed formula (1).

For the uniqueness we only note that the stated inner product formula determines all the entries of the matrices on each side (as we have used at the start of this proof).  $\square$

**Remark.** The formula (1) is often used as a definition of transpose, the description by swapping rows for columns being then a consequence.

Now, we will use the inner product formula (1) to deduce our desired properties of the transpose.

*Proof.* (theorem 1) The first linearity property of the transpose is a consequence of the linearity of inner product:

$$\begin{aligned} [(aA + bB)\mathbf{x}] \cdot \mathbf{y} &= (aA\mathbf{x} + bB\mathbf{x}) \cdot \mathbf{y} = aA\mathbf{x} \cdot \mathbf{y} + bB\mathbf{x} \cdot \mathbf{y} = \\ &= a\mathbf{x} \cdot A^T\mathbf{y} + b\mathbf{x} \cdot B^T\mathbf{y} = \mathbf{x} \cdot (aA^T\mathbf{y} + bB^T\mathbf{y}) = \mathbf{x} \cdot [(aA^T + bB^T)\mathbf{y}] \end{aligned}$$

so that, by the uniqueness in prop. 2, we have  $(aA + bB)^T = aA^T + bB^T$ . For the 2nd property we have, for any  $\mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{R}^l$  that

$$(CA\mathbf{x}) \cdot \mathbf{z} = (A\mathbf{x}) \cdot (C^T\mathbf{z}) = \mathbf{x} \cdot (A^T C^T \mathbf{z})$$

so that, by the uniqueness in prop. 2, we have  $(CA)^T = A^T C^T$ . The 3rd property, is a consequence of  $I^T = I$  and using that  $A$  being invertible satisfies  $AA^{-1} = I = A^{-1}A$ . Taking the transpose of this last equation we have (with property 2):

$$(A^{-1})^T A^T = I^T = I = A^T (A^{-1})^T$$

so that  $A^T$  is invertible with  $(A^T)^{-1} = (A^{-1})^T$ .  $\square$

*Proof.* (theorem 2) It is useful to recall that given some vector  $\mathbf{v} \in \mathbb{R}^n$  then  $\mathbf{v} = \mathbf{0} \iff \mathbf{v} \cdot \mathbf{x} = 0, \forall \mathbf{x} \in \mathbb{R}^n$  (this is usually called ‘non-degeneracy’ of the inner product).

With this then, for the first item, we have:

$$\begin{aligned} \mathbf{x} \in \ker A &\iff A\mathbf{x} = \mathbf{0} \iff A\mathbf{x} \cdot \mathbf{y} = 0, \forall \mathbf{y} \in \mathbb{R}^m \\ &\iff \mathbf{x} \cdot A^T\mathbf{y} = 0, \forall \mathbf{y} \in \mathbb{R}^m \iff \mathbf{x} \in (\text{im}(A^T))^\perp. \end{aligned}$$

The 2nd item follows from the first, applied to  $A^T$  and using that  $(A^T)^T = A$ :  $\ker A^T \stackrel{(1)}{=} (\text{im}((A^T)^T)^\perp \stackrel{(A^T)^T=A}{=} (\text{im}(A))^\perp$ .  $\square$

**Remark.** We have seen for example how the properties in theorem 2 can be useful to justify efficient computations to extract e.g. a system of formulas for the  $\text{im}(A)$  as a level set (by reducing  $A^T$  to FER form).

EXERCISE.

- Deduce from theorem 2 that  $\text{rank}(A) = \text{rank}(A^T)$  (remember rank-nullity theorem). Here you may use that for any subspace  $U \subset \mathbb{R}^k$  then  $\dim U + \dim U^\perp = k$ .