Integral calculus (review and some extra themes for practice)

Contents

We present some applications to practice more with the general Stokes' theorem (ie: Green, Stokes, Gauss theorems) taking the general type of form:

$$\int_{\partial D} \omega = \int_D d\omega$$

(see §A below for a summary of these main formulas).

1 a complex variable

Consider the plane $\mathbb{R}^2$ identified in the usual way with complex numbers $\mathbb{C} \cong \mathbb{R}^2$:

$$\mathbb{R}^2 \ni (x, y) \longleftrightarrow x + iy = z \in \mathbb{C}.$$

Under this identification we may consider $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (u(x, y), v(x, y))$ equally well as a map

$$f : \mathbb{C} \to \mathbb{C}, \quad z = x + iy \mapsto u(x, y) + iv(x, y) = f(z)$$

and vice-versa.

Exercise. Let $C \subset \mathbb{R}^2 \cong \mathbb{C}$ be a regular curve and $f : \mathbb{C} \to \mathbb{C}$. Consider[1]:

$$\int_C f dz := \int_{t_0}^{t_1} f(z(t))z'(t)\ dt \in \mathbb{C}$$

for $[t_0, t_1] \ni t \mapsto z(t) = x(t) + iy(t)$ a parametrization of C. Writing $f(z) = u(x, y) + iv(x, y)$ as above, expand to obtain:

$$\int_C f dz = \int_C u dx - v dy + i \int_C v dx + u dy.$$

Exercise. Suppose $f = u + iv : \mathbb{C} \to \mathbb{C}$ has $u(x, y), v(x, y)$ *of class* C^1 over the whole plane. Use Green's theorem (eq. (2)) to show that

$$\oint_C f dz = 0$$

for any regular loop $C \subset \mathbb{C}$ iff $u(x, y), v(x, y)$ satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Exercise. Consider the function: $f : \mathbb{C}\backslash 0 \to \mathbb{C}\backslash 0,\ z \mapsto \frac{1}{z}$, and let $C \subset \mathbb{C}\backslash 0$ be a regular loop not passing through the origin. Describe how the value of:

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z}$$

depends on the loop C (remember winding number!).

[1] The integral of a complex valued function $F = U + iV : \mathbb{R} \to \mathbb{C}$, by definition, being extended by linearity: $\int F\ dt = \int U dt + i \int V dt$.

2 harmonic functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ (say for $n = 2$ or 3) of class C^2 is called *harmonic* at those points where

$$0 = \Delta f = \mathrm{div}(\mathrm{grad}(f)) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + ...$$

(the operator $f \mapsto \Delta f$ is called the *Laplacian operator*). In what follows, let $R \subset \mathbb{R}^n$ be a bounded and path connected domain with smooth boundary ∂R.

Exercise. Use the divergence theorem (eq. (3)) to establish the following integration by parts formula:

$$\int_R f \mathrm{div}(\mathbf{v}) \; dV = \int_{\partial R} f \mathbf{v} \cdot d\mathbf{A} - \int_R \nabla f \cdot \mathbf{v} \; dV$$

for f and $\mathbf{v}$ of class C^1 around R. Note: it follows from product rule that $\mathrm{div}(f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f \mathrm{div}(\mathbf{v})$.

Exercise. Apply the integration by parts formula from the last exercise to establish the following Green's formulas:

$$\int_R f \Delta g \; dV = \int_{\partial R} f \frac{\partial g}{\partial n} \; dA - \int_R \nabla f \cdot \nabla g \; dV$$

$$\int_R f \Delta g - g \Delta f \; dV = \int_{\partial R} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \; dA$$

for f, g of class C^2 around R, and where $\frac{\partial f}{\partial n} := \nabla f \cdot N$ is the *normal derivative* of f along ∂R (for N the exterior unit normal to ∂R).

Exercise. Suppose f is harmonic over R. From the last exercise, deduce that $\int_R |\nabla f|^2 dV = \int_{\partial R} f \frac{\partial f}{\partial n} \; dA$. Conclude: if $f|_{\partial R} = 0$ then $f = 0$ over R (likewise if $\frac{\partial f}{\partial n} = 0$ over ∂R then $f = cst.$ over R).
Note: if $\nabla f = 0$ over a path connected region R then $f = cst.$ over R (prove it!).

Exercise. Show that if f and g are harmonic over R and $f|_{\partial R} = g|_{\partial R}$ then $f = g$ over R (apply the last exercise to $h = f - g$).

Remark. *This last exercise establishes* uniqueness *of solutions (when they exist) to the p.d.e. $\Delta f = 0$ on R and $f|_{\partial R} = f_0$ for given boundary condition $f_0 : \partial R \to \mathbb{R}$. Similarly we have uniqueness upto shifts by a constant of solutions (when they exist) to the p.d.e. $\Delta f = 0$ on R and $\frac{\partial f}{\partial n} = f_1$ for given (Neumann) boundary condition $f_1 : \partial R \to \mathbb{R}$. This is an important fact when considering static electric fields produced by given conducting surfaces in the presence of (fixed) ambient charges.*

In the next series of exercises we will establish the important *mean value property* of harmonic functions. Namely, for f of class C^2 and harmonic over some ball surrounding a given point $p \in \mathbb{R}^n$ we have:

$$f(p) = \frac{\int_{S_r(p)} f\ dA}{Area(S_r(p))} = \frac{\int_{B_r(p)} f\ dV}{Vol(B_r(p))} \tag{1}$$

whenever $\Delta f = 0$ over $B_r(p)$ for $B_r(p)$ a solid ball centered at p of radius r and $S_r(p) = \partial B_r(p)$ a sphere centered at p of radius r. In the following exercises we will re-center our coordinates at p to consider without loss of generality the case when f is harmonic over some ball around the origin.

Exercise. Consider the function (say for $n = 2$ or $n = 3$):

$$v : \mathbb{R}^n \backslash 0 \to \mathbb{R}, \quad \mathbf{x} \mapsto \begin{cases} \frac{1}{(2-n)|\mathbf{x}|^{n-2}}, & n \neq 2, \\ \frac{1}{2}\log|\mathbf{x}|, & n = 2. \end{cases}$$

Show that $\mathrm{grad}(v)|_{\mathbf{x}} = \nabla v|_{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^n}$ and $\Delta(v) = \mathrm{div}(\mathrm{grad}(v)) = 0$ over $\mathbb{R}^n \backslash 0$.

Exercise.* Let $R \subset \mathbb{R}^n$ be some solid region containing the origin and $f : \mathbb{R}^n \to \mathbb{R}$ of class C^2 over R. Take $R_\epsilon = R \backslash B_\epsilon$ for $B_\epsilon \subset R$ a ball centered at the origin of radius $\epsilon > 0$ (small enough to be contained within R). Apply Green's formula (from the previous page) to f, v over R_ϵ to deduce that:

$$\int_{R_\epsilon} v \Delta f\ dV = \int_{\partial R} v\frac{\partial f}{\partial n} - f\frac{\partial v}{\partial n}\ dA + \int_{\partial B_\epsilon} f\frac{\partial v}{\partial n}\ dA - \frac{\epsilon^{2-n}}{2-n}\int_{B_\epsilon} \Delta f\ dV$$

Using that f is of class C^2, so we can write $f(\mathbf{x}) = f(0) + o(|\mathbf{x}|)$, $(\Delta f)(\mathbf{x}) = \Delta f(0) + o(|\mathbf{x}|)$ where $\lim_{|\mathbf{x}| \to 0} o(|\mathbf{x}|) = 0$. Let $\epsilon \to 0$ to conclude:

$$(*) \qquad \int_R v\Delta f\ dV = \int_{\partial R} \left(v\frac{\partial f}{\partial n} - f\frac{\partial v}{\partial n} \right)\ dA + f(0)Area(S_1(0))$$

where $S_1(0) = \partial B_1(0)$ is the unit sphere centered at the origin.

Exercise. Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and vanishing outside some ball around the origin. Set $\bar{v} = \frac{v}{Area(S_1(0))}$ and deduce from the last exercise $(*)$ that:

$$\int_{\mathbb{R}^n} \bar{v}\ \Delta f\ dV = f(0).$$

Remark. *This last exercise establishes that $\bar{v}$ is what is called a* fundamental solution *for the Laplacian operator. In particular it can be used to generate formulas (by convolution) for solutions to the Poisson equation $\Delta u = \rho$.*

Exercise. Let $f : \mathbb{R}^n \to \mathbb{R}$ be harmonic over some ball $B_r(0)$ around the origin and apply formula $(*)$ above with $R = B_r(0)$ to conclude the mean value property eq. (1) above (with $p = 0$). Show that the mean value property implies the *maximum principle*: if f is harmonic over R then the maximum of f occurs at some point of ∂R.

3 operators in different coordinates

We have typically written formulas in usual Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$. It is often useful to compute formulas for say the fundamental gradient/curl/divergence/Laplacian operators in some other coordinates. For *orthogonal* coordinates we may determine these formulas by taking advantage of our integral theorems. First:

Definition 1. *We call a smooth locally invertible map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3,\ (u, v, w) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w)) = \Phi(u, v, w)$ an* orthogonal *system of coordinates when*

$$\frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} = \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial w} = \frac{\partial \Phi}{\partial w} \cdot \frac{\partial \Phi}{\partial u} = 0$$

and we set $a := |\frac{\partial \Phi}{\partial u}| \neq 0,\ b := |\frac{\partial \Phi}{\partial v}| \neq 0,\ c := |\frac{\partial \Phi}{\partial w}| \neq 0$.

Exercise. Verify that polar (or cylindrical) coordinates are orthogonal coordinates, as are spherical coordinates.

Exercise. Another type of orthogonal coordinate system are *elliptic coordinates*. We describe them in the plane $\mathbb{R}^2$:

1. Fix two points say at $\mathbf{f}_1 = (-1, 0), \mathbf{f}_2 = (1, 0) \in \mathbb{R}^2$. Show that a conic section having foci at $\mathbf{f}_j$ is given by a point locus of the form:

$$\frac{x^2}{\mu} + \frac{y^2}{\mu - 1} = 1,\ (\mu > 1;\ \textit{ellipse}), \quad \frac{x^2}{\nu} - \frac{y^2}{1 - \nu} = 1, \quad (\nu \in (0, 1);\ \textit{hyperbola}).$$

2. Parametrize points in the quadrant $x > 0, y > 0$ by sending $\mu \in (1, \infty), \nu \in (0, 1)$ to the intersection point in said quadrant of the ellipse $\frac{x^2}{\mu} + \frac{y^2}{\mu - 1} = 1$ and hyperbola $\frac{x^2}{\nu} - \frac{y^2}{1-\nu} = 1$. Show that we then have:

$$x^2 = \mu\nu, \quad y^2 = (\mu - 1)(1 - \nu).$$

3. Show elliptic coordinates are an orthogonal system of coordinates (there is a direct analytic proof, and also a geometric proof).

Remark. *Elliptic coordinates can be generalized to higher dimensions by considering to some given constants $A > B > C > 0$ the parametrization of octants of $\mathbb{R}^3$ by sending $(u, v, w) \in (-\infty, C) \times (C, B) \times (B, A)$ to the intersection points of the quadratic surfaces defined by the implicit equations:*

$$\frac{x^2}{A-u} + \frac{y^2}{B-u} + \frac{z^2}{C-u} = 1, \quad \frac{x^2}{A-v} + \frac{y^2}{B-v} + \frac{z^2}{C-v} = 1, \quad \frac{x^2}{A-w} + \frac{y^2}{B-w} + \frac{z^2}{C-w} = 1$$

These intersection points are, explicitly:

$$x^2 = \frac{(A-u)(A-v)(A-w)}{(A-B)(A-C)}, \quad y^2 = \frac{(B-u)(B-v)(B-w)}{(B-A)(B-C)}, \quad z^2 = \frac{(C-u)(C-v)(C-w)}{(C-B)(C-A)}.$$

Having now some examples of orthogonal coordinates, let $\Phi(u,v,w)$ be some orthogonal coordinates and set $a = |\frac{\partial \Phi}{\partial u}|,\ b = |\frac{\partial \Phi}{\partial v}|,\ c = |\frac{\partial \Phi}{\partial w}|$ as well as

$$\mathbf{e}_u = \frac{1}{a}\frac{\partial \Phi}{\partial u},\ \mathbf{e}_v = \frac{1}{b}\frac{\partial \Phi}{\partial v},\ \mathbf{e}_w = \frac{1}{c}\frac{\partial \Phi}{\partial w}.$$

Exercise. Let $f : \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto f(x,y,z)$ be of class C^1. Show that the gradient of f: $\nabla f|_p \cdot \vec{v} = d_p f(\vec{v})$ is given in the (orthonormal) basis $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$ by:

$$\nabla f = \frac{1}{a}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{b}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{c}\frac{\partial f}{\partial w}\mathbf{e}_w$$

(where we write for short $\frac{\partial f}{\partial u}$ for $\frac{\partial (f\circ\Phi)}{\partial u}$).

Exercise.* Let $\mathbf{X} : \mathbb{R}^3 \to \mathbb{R}^3,\ (x,y,z) \mapsto \mathbf{X}(x,y,z)$ be of class C^1 and write

$$\mathbf{X} = X^u\mathbf{e}_u + X^v\mathbf{e}_v + X^w\mathbf{e}_w$$

in the above orthonormal basis. Consider the image of a (small) coordinate cube: $\Phi(C_\epsilon)$ where

$$C_\epsilon = \{u_o \le u \le u_o + \epsilon,\ v_o \le v \le v_o + \epsilon,\ w_o \le w \le w_o + \epsilon\}.$$

By the divergence theorem eq. (3), we have: $\int_{\Phi(C_\epsilon)} \operatorname{div}(\mathbf{X})\ dV = \int_{\partial\Phi(C_\epsilon)} \mathbf{X}\cdot d\mathbf{A}$. Show that these last integrals can be written:

$$\int_{C_\epsilon} abc \operatorname{div}(\mathbf{X})\ dudvdw = \int_{C_\epsilon} \frac{\partial}{\partial u}(bcX^u) + \frac{\partial}{\partial v}(caX^v) + \frac{\partial}{\partial w}(abX^w)\ dudvdw.$$

Since the last equation holds for any cube C_ϵ and our integrands are (by assumption) continuous deduce that:

$$(*) \qquad abc \operatorname{div}(\mathbf{X}) = \frac{\partial}{\partial u}(bcX^u) + \frac{\partial}{\partial v}(caX^v) + \frac{\partial}{\partial w}(abX^w).$$

Exercise. Conclude from the last two exercises that for $f : \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto f(x,y,z)$ of class C^2 we have:

$$abc\ \Delta f = \frac{\partial}{\partial u}\left(\frac{bc}{a}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{ca}{b}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{ab}{c}\frac{\partial f}{\partial w}\right)$$

Remark. *Similarly we find the curl of $\mathbf{X} = X^u\mathbf{e}_u + X^v\mathbf{e}_v + X^w\mathbf{e}_w$ is given in orthogonal coordinates by:*

$$abc\ \nabla\times\mathbf{X} = \det\begin{pmatrix} a\mathbf{e}_u & b\mathbf{e}_v & c\mathbf{e}_w \\ \partial_u & \partial_v & \partial_w \\ aX^u & bX^v & cX^w \end{pmatrix}.$$

Note: our various types of integrals are $\int_C \mathbf{X}\cdot d\mathbf{s} = \int_C aX^u du + bX^v dv + cX^w dw$, and $\int_S \mathbf{X}\cdot d\mathbf{A} = \int_S bcX^u\ dv\wedge dw + caX^v dw\wedge du + abX^w du\wedge dv$, and $\int_{\Phi(U)} \rho\ dV = \int_U abc\rho\ dudvdw$ in such (oriented) orthogonal coordinates.

A Review (main formulas)

We have studied the following integral formulas (generalizations of the fundamental theorem of calculus to higher dimensions):

- Green theorem:

$$\oint_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{2}$$

  for $D \subset \mathbb{R}^2$ a bounded region enclosed by a piecewise C^1 boundary curve ∂D and $(P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1 around D. The line integral on the left should be oriented with unit tangent T so that the rotation of T by $\pi/2$ counterclockwise is inwards pointing to D.

- Divergence (Gauss) theorem:

$$\oint_{\partial D} Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy = \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz \tag{3}$$

  for $D \subset \mathbb{R}^3$ a bounded region enclosed by a piecewise C^1 boundary surface ∂D and $(P,Q,R) : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^1 around D. The surface integral on the left should be oriented with unit normal pointing outwards of D.

- Stokes theorem:

$$\oint_{\partial D} Pdx+Qdy+Rdz = \int_D \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy \tag{4}$$

  for $D \subset \mathbb{R}^3$ a bounded surface enclosed by a piecewise C^1 boundary curve ∂D and $(P,Q,R) : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^1 around D. The line integral and surface integral are oriented so that the unit tangent T along ∂D and unit normal N along D have $N \wedge T$ inwards pointing along D.

- We should also not forget that for line integrals of gradient fields (also called conservative fields) we have:

$$\int_C df = f(\mathbf{b}) - f(\mathbf{a})$$

  where C is an arc from point $\mathbf{a}$ to point $\mathbf{b}$ and $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + ...$

Remark. *The notation above (differential forms notation/Leibniz notation) is meant to remind us how to compute the integrals. Namely for a line integral of $\mathbf{v}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ over a curve $C = im(\gamma)$ parametrized by $[t_0,t_1] = dom(\gamma) \ni t \mapsto \gamma(t) = (x(t), y(t), z(t))$ then:*

$$\int_C Pdx + Qdy + Rdz = \int_{t_0}^{t_1} P(\gamma(t))x'(t) + Q(\gamma(t))y'(t) + R(\gamma(t))z'(t)\ dt$$

Likewise for a surface integral of $\mathbf{v}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ *over a surface* $S = im(\varphi)$ *parametrized by* $U \ni (u,v) \mapsto \varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, *for* $U = dom(\varphi) \subset \mathbb{R}^2$ *the relevant domain of* φ, *then:*

$$\int_S P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy =$$

$$= \int_U P(\varphi(u,v))\frac{\partial(y,z)}{\partial(u,v)} + Q(\varphi(u,v))\frac{\partial(z,x)}{\partial(u,v)} + R(\varphi(u,v))\frac{\partial(x,y)}{\partial(u,v)}\ dudv$$

where $\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{pmatrix} y_u & y_v \\ z_u & z_v \end{pmatrix}$, *...etc. (note* $dy \wedge dz = (y_u du + y_v dv) \wedge (z_u du + z_v dv) = (y_u z_v - y_v z_u) du \wedge dv$, *...etc.) Some other common ways to write line and surface integrals are:*

$$\int_C \mathbf{v} \cdot T\ ds = \int_C \mathbf{v} \cdot d\mathbf{s} = \int_C P dx + Q dy + R dz$$

$$\int_S \mathbf{v} \cdot N\ dA = \int_S \mathbf{v} \cdot d\mathbf{A} = \int_S P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$$

for $\mathbf{v} = (P, Q, R)$ *in components.*

Remark. *The partial derivatives appearing in the divergence theorem and Stokes theorem are also commonly denoted by:*

$$div(\mathbf{v}) = \nabla \cdot \mathbf{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$curl(\mathbf{v}) = rot(\mathbf{v}) = \nabla \times \mathbf{v} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

for $\mathbf{v} = (P, Q, R)$ *in components (the last notation is meant to remind you how to compute them, by setting* $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$*). Our integral theorems then read as:*

$$\oint_{\partial D} \mathbf{v} \cdot d\mathbf{s} = \int_D (\nabla \times \mathbf{v}) \cdot d\mathbf{A},$$

$$\oint_{\partial D} \mathbf{v} \cdot d\mathbf{A} = \int_D (\nabla \cdot \mathbf{v})\ dV.$$

Remark. *While the curl and divergence operate on vector fields, we also should not forget the gradient operation on functions:*

$$grad(f) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

related to the differential of $f : \mathbb{R}^3 \to \mathbb{R}$ *through:*

$$df(\vec{v}) = (\nabla f) \cdot \vec{v}$$

(at each given point the differential of f *is a linear map* $d_p f : \mathbb{R}^3 \to \mathbb{R}$*).*