# CLASSICAL MECHANICS NOTES[1]



Classical mechanics, as a branch of physics, studies the motion of objects. The model is fundamental in applications when relativistic or quantum effects may be ignored. In fact, a main 'test' of such relativistic or quantum theories is that they recover classical mechanics in some appropriate limit.

Mathematically, these descriptions of motion are given by differential equations. Broadly speaking then, one may think of two fundamental goals in being able to apply the theory in a useful way. First, one should have general principles from which one can derive the differential equations governing the motions of the system. Secondly, once the relevant differential equations are in hand, one would like some methods to describe the properties of its solutions: ie make predictions of how the system will behave.

In these course notes we consider addressing these two goals in the 'Newtonian, Lagrangian and Hamiltonian' frameworks[2]. We will consider in detail various examples, such as rigid bodies and problems from celestial mechanics.


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[2]Following Arnold's *Mathematical methods of classical mechanics* as our principal reference.

# I. Newtonian mechanics

## §1: reference frames[1]

The motion of an object is described by giving its position measurements as functions of time. As position and time are relative concepts, so too is motion, depending on the coordinate system or *reference frame* in which it is described.

Figure 1. Position and time measurements are *relative*, meaning they have sense only in reference to other objects/events. For example, we give years in A.D./B.C. and in the figure above the house is to the left/right of the road depending on whether we direct ourselves along the road towards the trees/mountains.

For example a point particle in motion is described in a system of coordinates as a curve in $\mathbb{R}^3$.

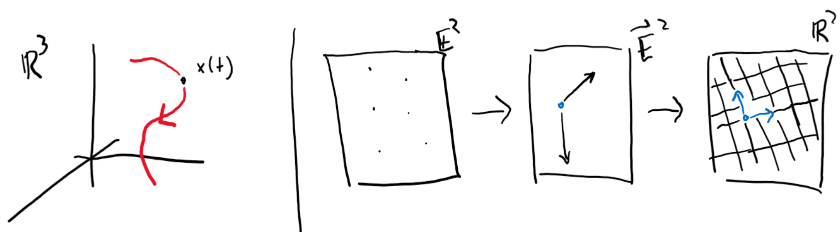


Figure 2. In a system of coordinates, the motion of a particle is given by a curve $x : \mathbb{R} \to \mathbb{R}^3, t \mapsto x(t)$. The choice of coordinate system for space requires specifying an origin and coordinate axes (orthonormal basis) at each instant.

Such a coordinate description depends on the following *choices*:

* at each instant of time, an origin in space,

* at each instant of time a system of coordinate axes based at said origin,

* choice of an origin and unit of time.

This freedom in choice of reference frame is, in practice, an advantage as one may choose coordinates well adapted to the problem making it simpler to analyze. Mathematically, space at each instant is assumed to be an (Euclidean) affine space:[2]

$$\mathbb{E}^3 \overset{origin}{\to} \vec{\mathbb{E}}^3 \overset{basis}{\to} \mathbb{R}^3.$$

[1]The explanation of reference frames in Marle's lectures: C. Marle, *Cours de mecanique*. Universite Pierre et Marie Curie, Paris VI, (1978-79). Available online here, is very thorough and to the point.

[2]Intuitively an affine space is a 'vector space without a specified origin'. More precisely, it is a set on which a vector space $V$ acts freely and transitively, although we will not need this definition here –the intuitive image of a vector space upto choice of origin sufficing. By Euclidean affine spaces we mean that the underlying vector space is equipped with an inner product, and consider orthonormal bases giving our reference frames.

The dependence of such choices on time may be visualized in the *space-time*, $\mathcal{M} := \sqcup_{\tau\in\mathcal{T}} \mathbb{E}^3_\tau$, where $\mathcal{T}$ is *the* (universal) time-line and for each $\tau \in \mathcal{T}$, $\mathbb{E}^3_\tau$ is the affine space of possible positions at the instant $\tau$.

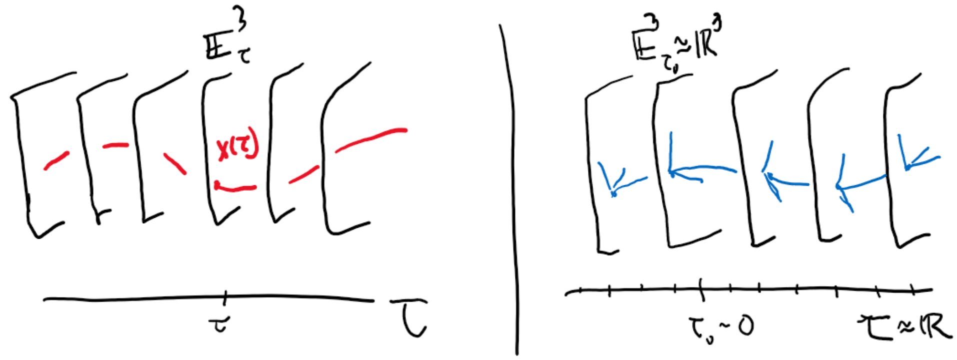


Figure 3. Space-time, $\mathcal{M}$, is the set of all possible positions, $\mathbb{E}^3_\tau$, at given instants, $\tau \in \mathcal{T}$. Each particle has a 'worldline' (left) in the space time, and the choice of a reference frame (right) is given by choosing origin and bases for space at each time (as well as an origin, $\tau_o$, and unit of time allowing the identification $\mathcal{T} \cong \mathbb{R}$).

To take advantage of the freedom in choice of frame, it is necessary to become adept at performing 'changes of variables' resulting from such shifts in choice of reference frame. Given a reference frame $(x,t) \in \mathbb{R}^3 \times \mathbb{R} \cong \mathcal{M}$, *any* other reference frame is related to it through a transformation of the form:

$$\mathbb{R}^3 \times \mathbb{R} \ni (\tilde{x}, \tilde{t}) = (A(t)x + b(t), ct + d)$$

where $A(t) \in \mathrm{O}_3, b(t) \in \mathbb{R}^3, c \in \mathbb{R}_{>0}, d \in \mathbb{R}$. Velocities and accelerations of motions in the $(x,t)$-frame may be related to those measured in the $(\tilde{x}, \tilde{t})$-frame by differentiation (chain rule/product rules). Of particular importance are the following two cases:

* uniform translations: $(\tilde{x}, \tilde{t}) = (x + at, t)$ for $a \in \mathbb{R}^3$,

* uniform rotations: $(\tilde{x}, \tilde{t}) = (R_{\vec{\omega}}(t)x, t)$ for $R_{\vec{\omega}}(t) \in \mathrm{SO}_3$ rotation by angle $|\vec{\omega}|t$ around the $\vec{\omega} \in \mathbb{R}^3$ axis.

Additionally there is the possibility to choose general (not necessarily linear) coordinates: $\mathbb{R}^3 \times \mathbb{R} \ni (y,s) = (f(x,t), g(t))$ for $(x,t)$ coordinates from some reference frame and $f(\cdot,t) : V \subset \mathbb{R}^3 \to U \subset \mathbb{R}^3, g : \mathbb{R} \to \mathbb{R}$ diffeomorphisms. The most relevant examples are:

* cylindrical coordinates: $(r, \theta, z, t) \to (r\cos\theta, r\sin\theta, z, t)$

* spherical coordinates: $(\rho, \varphi, \theta, t) \to (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi, t)$.

And we will meet as well some other general coordinate systems in the course (eg elliptical coordinates).

Remark: The changes of variable above are very similar to those in multivariable calculus, with the added possibility that the new coordinate systems depend on time. This extra condition is relevant in practice when, for example, observing the stars from the surface of the earth (approximately a uniformly rotating coordinate system with respect to a coordinate system 'fixed to the stars').

Exercises:

1. Consider on the plane a system of coordinates, $\tilde{q}$, rotating uniformly counterclockwise with respect to the system of coordinates $q$ with angular velocity $\omega$. Show that a point particle $q(t)$ in motion in the first coordinate system has acceleration in the second coordinate system given by (using the identification $\mathbb{R}^2 = \mathbb{C}$):
$$\ddot{\tilde{q}}(t) = -2i\omega\dot{\tilde{q}} + \omega^2\tilde{q} + e^{-i\omega t}\ddot{q}.$$

2. The orthogonal group (or rotation group) –denoted by $\mathrm{O}_3$– are linear maps $A : \mathbb{R}^3 \to \mathbb{R}^3$ which preserve the dot product: $A\vec{u} \cdot A\vec{v} = \vec{u} \cdot \vec{v}, \forall \vec{u}, \vec{v} \in \mathbb{R}^3$.

   (a) Show that $A \in \mathrm{O}_3 \iff A^T A = I$. Deduce that $\det A = \pm 1$.

   Orientation preserving elements of $\mathrm{O}_3$ (that is those with determinant 1) are denoted $\mathrm{SO}_3$.

   (b) Let $\mathbb{R} \ni t \mapsto A(t) \in \mathrm{SO}_3$ be a smooth curve of rotations with $A(0) = I$. Show that $\Omega := \frac{d}{dt}|_{t=0}A(t)$ is a skew-symmetric linear map ($\Omega^T = -\Omega$).

   The set of skew-symmetric linear maps are denoted by $\mathfrak{so}_3$.

   (c) For $\vec{\omega} \in \mathbb{R}^3$ consider $\Omega_{\vec{\omega}} : \mathbb{R}^3 \to \mathbb{R}^3, \vec{v} \mapsto \vec{\omega} \times \vec{v}$. Show that $\Omega_{\vec{\omega}} \in \mathfrak{so}_3$ and $\mathbb{R}^3 \to \mathfrak{so}_3, \vec{\omega} \mapsto \Omega_{\vec{\omega}}$ is a vector space isomorphism.

   We call $\vec{\omega}$ an *infinitesimal rotation axis* and $\Omega_{\vec{\omega}}$ an infinitesimal rotation.

3. Let $A(t) \in \mathrm{SO}_3$ be rotation by angle $\omega t$ around the $\hat{k}$ ($z$)-axis. Show that
$$\frac{d}{dt}|_{t=s}A(t)A(s)^{-1} = \frac{d}{dt}|_{t=s}A(s)^{-1}A(t) = \Omega_{\vec{\omega}}$$
   where $\vec{\omega} = \omega\hat{k}$.

4. For $A(t)$ as in # 3, let $A(t)\vec{y} = \vec{x}$ be a uniformly rotating coordinate system. Show that:
$$\ddot{\vec{y}} = -2\vec{\omega} \times \dot{\vec{y}} - \vec{\omega} \times (\vec{\omega} \times \vec{y}) + A^{-1}\ddot{\vec{x}}.$$

5. Consider a parametrized curve given in Cartesian coordinates by $t \mapsto (x(t), y(t), z(t)) \in \mathbb{R}^3$.

   (a) Express the norm of velocity squared, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, in cylindrical coordinates.

   (b) Express the norm of velocity squared, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, in spherical coordinates.

6. Consider a particle $t \mapsto \vec{q}(t)$ moving over the surface of a sphere of radius $R$ (so say $|\vec{q}(t)| = R$). At each instant of time, the acceleration may be divided into its tangential and normal components: $\ddot{\vec{q}} = \ddot{\vec{q}}_{tan} + \ddot{\vec{q}}_{nor}$ where $\ddot{\vec{q}}_{tan}, \ddot{\vec{q}}_{nor}$ are tangent and normal to the sphere at $q(t)$.

   (a) Show that $\ddot{\vec{q}}_{nor} = -\frac{|\dot{\vec{q}}|^2}{R^2}\vec{q}$.

   (b) The motion of $\vec{q}(t)$ on the sphere is said to be *free* if its tangential component vanishes. Determine the free motions on the sphere.

   (c) For a motion on the sphere with unit velocity ($|\dot{\vec{q}}| = 1$), its geodesic curvature is defined as $\kappa_{geo} := |\ddot{\vec{q}}_{tan}|$. Calculate the geodesic curvature of a latitude of the sphere.

7. The height, $h$, of a falling object ($\dot{h}(0) \leq 0$) subject to air resistance may be modeled by the ode: $\ddot{h} = -g + \gamma\dot{h}^2$, where $g, \gamma > 0$ are constants. Show that as $t \to \infty$, we have $\dot{h}(t) \to -\sqrt{g/\gamma}$.

### §2 laws of motion

The analysis of an objects motion is based on the following notions and language. First, an objects *natural* or *free* motion is understood as the motion the object *would* take in the absence of *any* external influences. Deviations in an objects motion from such free motions are then said to be caused by *forces* having been applied to the object. The tendency of an object to continue on a new trajectory after such forces have been applied is said to be due to its having acquired *momentum* as a result of the application of said forces.


Figure 4. The ideas may be illustrated by an imagined thrown ball. Initially the ball is at rest and if let go falls straight down. After exerting a force to throw the ball, the ball continues on a new trajectory as a result of its acquired momentum (in particular when we stop applying the force with our hand, the ball does not simply fall straight down as it would have initially).

The dynamical part of mechanics consists in making predictions about the resulting motions of an object when subject to given forces. For this to be useful and feasible requires more precision, ie defining what are the free motions, etc. These definitions form the content of the following axioms or 'laws of motion'.

From the previous section we have seen that motion is a relative notion: depending upon the reference frame taken. The description of free motions will thus of necessity include certain reference frames in which such free motions have a standard description. Namely, a reference frame is said to be 'stationary' or an *inertial frame* when its origin and axes are free from the influence of any external forces.

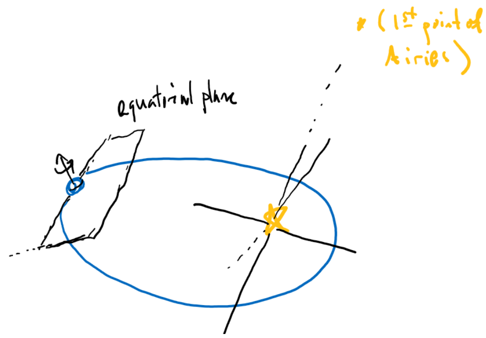


Figure 5. An example of an approximately 'stationary' or inertial frame is one 'fixed to the stars'. The stars are so massive and distant from eachother, that they may to very near approximation be considered 'fixed' or free from external influences. The origin is taken at the center of the sun and the $xy$-plane the plane containing the Earth-Sun orbit. The $x$-axis may be taken as parallel to the line in which Earth's equatorial plane intersects the $xy$-plane (a line which in the time of early astronomers pointed in the direction to a star in the constellation of Aires, the 'first point of Aires').

We now may state:

**Newton's 1st law (N1):**[1] In an inertial frame of reference, the free motion of a particle is of uniform velocity along a straight line. In formulas: free motions are those with $\frac{d^2x}{dt^2} = 0$, ie $x(t) = x_o + tv_o, \;\; x_o, v_o \in \mathbb{R}^3$.

[1](In an inertial frame) every object perseveres in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed thereon.

The geometric content of the existence of inertial frames and N1 is that $\mathcal{M}$ is equipped with an affine structure (the lines of which are free motions).

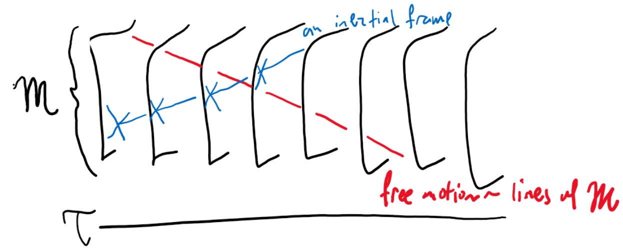


Figure 6. Free motions and inertial frames in space-time.

By definition, a force causes an object to deviate from its free motion (straight line in an inertial frame). In an inertial frame a force acting on a particle thus effects the higher derivatives ($\frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \ldots$) of an object.

**Newton's 2nd law (N2):**[1] The change in velocity (acceleration) of a particle in an inertial frame subject to a force is directly proportional to the force. This constant of proportionality is the *mass* of the object. In formulas: $f = m\ddot{x}$.

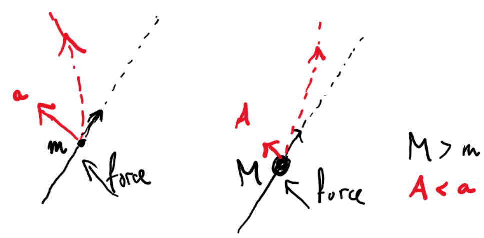


Figure 7. The *same* force applied to a more massive ('larger') object effects a smaller deviation (acceleration) from its free motion. N2 essentially defines mass as this 'resistance to change from linear motion' (also called the *linear inertia*, inertia referring to an objects resistance to change), in such a way that $MA = ma = f$.

Note that there is indeed content in N2 (it does not follow from N1), in the assumption that *all* forces causing deviations from the free motions of N1 are completely determined by only the *second* derivatives (and none of higher order) of the resulting deviating motion.

Furthermore, N2 essentially determines the momentum of a particle of mass $m$. As applying a force causes an object to acquire (change) momentum, and $f = m\ddot{x} = \frac{d}{dt}(m\dot{x})$, we define the (linear) *momentum* of a particle of mass $m$ moving with velocity $v = \dot{x}$ as:

$$p := mv = m\dot{x}.$$

The third (and final) law of motion imposes a condition on forces arising from interacting pairs of objects. It has a strong connection to the conserved quantities we will study in the next section:

**Newton's 3rd law (N3):**[2] Whenever object $A$ exerts a force $f$ on object $B$, then object $B$ exerts the force $-f$ on object $A$.

[1] The acceleration of altered motion is proportional to the force impressed; and directed along the same line.

[2] To every action there is an equal and opposite reaction; or, the mutual actions of two bodies upon eachother are always equal in strength and act in opposite directions.

Examples:

- For a particle of mass $m$ attached to a spring, we let $x$ be its displacement from the equilibrium position. Then $m\ddot{x} = f(x) = f'(0)x + O(x^2)$, since $f(0) = 0$ (when there is no displacement from equilibrium, the particle does not move). Since the spring exerts a restoring force (bringing the particle back towards equilibrium from its displacement) we have $f'(0) = -k < 0$. For small $x$, the behaviour is approximately that of $m\ddot{x} = -kx$ (Hooke's law), which has solutions $x(t) = A\cos(\omega t + \phi)$, where $\omega^2 = \frac{k}{m}$. The solutions may also be sketched in the position-velocity plane, where they are integral curves of the vector field $\dot{x} = v, \dot{v} = -\frac{k}{m}x$, lying on level sets of the function $\frac{mv^2}{2} + \frac{kx^2}{2}$.

- Approximating the Earth as an infinite plane (with coordinates as the $xy$-plane), a particle of mass $m$ is subject to the force of *uniform gravity*: $-mg\hat{k}$ (where $\hat{k} = (0, 0, 1)$). The equation of motion is $\ddot{x} = -g\hat{k}$ (Galileo's constant acceleration). Solutions follow parabolic arcs: $x(t) = x_o + v_o t - \frac{gt^2}{2}\hat{k}$.

- Consider a particle of mass $m$ attached to an end of a rigid rod which may rotate freely about its other end in a fixed plane. This is an example of a system with constraints: the point particle is only permitted to move along a fixed circle of radius $\ell$ (length of the rigid rod). We will see for such 'rotational motions' there are analogues of the above concepts (force, momentum, mass,...).

  Take coordinates centered at this circle of radius $\ell$ and let $\theta$ be the angular coordinate of the point mass $m$, so that with complex numbers the position of mass $m$ is given by $x = \ell e^{i\theta}$. Consider applying a force $f$ to the point mass $m$, how will the motion along the circle take place? In other words what equation of motion will govern the resulting $\theta(t)$?

  If $f||x$ then the force is entirely 'absorbed' by the rigidity of the rod: no motion will take place if $x$ was initially at rest. For a general force then, only its tangential component, $f_{tan} = f \cdot ix/\ell$, will effect the motion by influencing the tangential acceleration of $x$ according to N2: $m\ddot{x} \cdot ix = \ell f_{tan} = f \cdot ix$, or letting $\omega := \dot{\theta}$ be the *angular velocity*, $\alpha := \ddot{\theta}$ be the *angular acceleration*, $\tau := f \cdot ix$ the *torque*, $I := m\ell^2$ the *moment of Inertia*, we have the equation of motion:

$$\tau = I\alpha = \frac{d}{dt}I\omega = \frac{d}{dt}C$$

  where $C := I\omega$ is the *angular momentum.*

  These considerations may be applied to derive the equations of motion for a *circular pendulum*, when $f = -mg\hat{k}$ (uniform gravity), to obtain $m\ell^2\ddot{\theta} = -mg\ell\sin\theta$ (with $\theta$ the angle measured from the 'straight down' position).

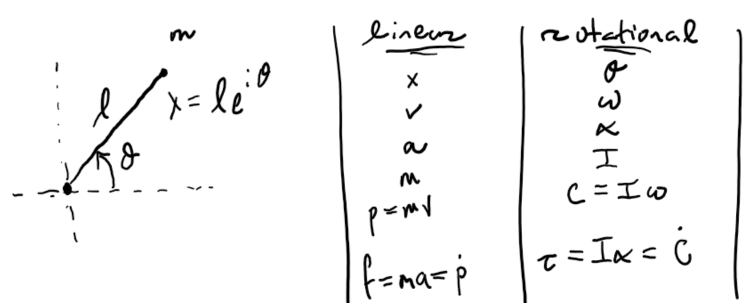


Figure 8. Rotational and linear analogues.

Exercises:

1. Sketch solutions of the pendulum equation: $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$ in the $(\theta, \dot{\theta})$-plane.

2. An involute of a plane curve is the set of points traced by the end of a cord of fixed length attached to the curve at a fixed point. Show that the tangent line to the involute of a plane curve is perpendicular to the cord at each instant.

3. The cycloid curve is defined as the curve traced by a point on a circle as the circle rolls along a line. Compute the length of an arc of the cycloid (the curve traced after the circle has completed one revolution), and determine the involute of a cycloid when a cord with half the length of an arc is attached to one of the cycloids 'cusps'.

4. Consider a particle, $\vec{x}$, of mass $m$ constrained to circular motion (centered at the origin) in a fixed plane inside $\mathbb{R}^3$. Show that:

   (a) $\dot{\vec{x}} = \vec{\omega} \times \vec{x}$ (where $\vec{\omega} = \omega\hat{n}$ for $\omega$ the angular velocity and $\hat{n}$ an appropriate unit normal to the fixed plane of motion).

   (b) $\vec{C} = m\vec{x} \times \dot{\vec{x}}$ (where $\vec{C} = C\hat{n}$ for $C = I\omega = m|\vec{x}|^2\omega$ the angular momentum of $\vec{x}$).

   (c) if $\vec{x}$ is subject to the force $\vec{f}$ (tangent to the fixed plane), then $\vec{\tau} = \vec{x} \times \vec{f}$ (where $\vec{\tau} = \tau\hat{n}$ for $\tau$ the torque due to $\vec{f}$ on $\vec{x}$).

5. For a general particle $\vec{x}(t) \in \mathbb{R}^3$ of mass $m$ moving in space, subject to forces $\vec{f}$, we define its angular momentum (with respect to the origin) as $\vec{C} := \vec{x} \times m\dot{\vec{x}}$ and torque (with respect to the origin) due to the forces $\vec{f}$ as $\vec{\tau} := \vec{x} \times \vec{f}$. Show that $\dot{\vec{C}} = \vec{\tau}$.

6. Define the *inertia tensor* of the particle $\vec{x} \in \mathbb{R}^3$ of mass $m$ as $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3, \vec{\omega} \mapsto \vec{C}$ where $\vec{C}$ is the angular momentum $\vec{x}$ would have if subject to the instantaneous rotation $\vec{\omega}$ (that is $\dot{\vec{x}} = \vec{\omega} \times \vec{x}$).

   (a) Show that $\mathbb{I}$ is linear and symmetric ($\mathbb{I}^T = \mathbb{I}$).

   (b) Show that $\mathbb{I}\vec{\omega} \cdot \vec{\omega} = m|\vec{\omega} \times \vec{x}|^2$.

   (c) Express $\mathbb{I}$ in matrix form wrt the standard basis of $\mathbb{R}^3$ (in which say $\vec{x} = (x, y, z)$).

**§3 conserved quantities**

The equations of motion for a system of particles $q_1, ..., q_N \in \mathbb{R}^3$ are a system of second order ODE's:

$$m_j \ddot{q}_j = f_j(q_k, \dot{q}_k, t)$$

or a vector field (system of first order ODE's):

$$\dot{q}_j = p_j/m_j, \quad \dot{p}_j = f_j(q_k, p_k/m_k, t), \quad \dot{t} = 1 \qquad (*)$$

on the (enormous!) $6N+1$-dimensional space $(q, p, t) \in \mathbb{R}^{6N+1}$. *Conserved quantities* or *first integrals* are functions $g(q, p, t)$ which remain constant over solutions to $(*)$. Geometrically, the trajectories are constrained to lie in the level sets of the function $g : \mathbb{R}^{6N+1} \to \mathbb{R}$, allowing us to reduce the dimensions under consideration by 1, and make some progress on understanding the behaviour of such trajectories. In fact, as we will see during the course, essentially the only mechanical systems which we can 'solve' (ie describe completely the solutions to) are those for which we can find 'enough' conserved quantities.

We now describe conserved quantities appearing in quite general *closed systems*: those for which the particles of the system are only subject to forces due to their mutual interactions. First:

*In a closed system, the total linear momentum is conserved.*

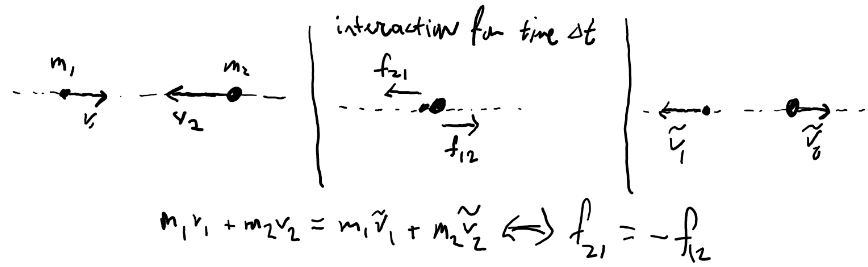


Figure 9. The conservation of linear momentum serves as a good motivation for N3. In fact, before Newton, this conservation was known experimentally for linear (elastic) collisions between particles, where it is equivalent to N3.

Indeed, let $q_1, ..., q_N \in \mathbb{R}^3$ be the particles of masses $m_j$ constituting the system. By N3, they satisfy:

$$m_j \ddot{q}_j = \sum_k f_{kj}$$

where $f_{kj} = -f_{jk}$ is the force on $q_j$ due to $q_k$. The *total linear momentum* of the system is:

$$p_{tot} := \sum_j p_j = \sum_j m_j \dot{q}_j,$$

and we have:

$$\dot{p}_{tot} = \sum_j \sum_k f_{kj} = 0$$

since by N3, the forces pairwise cancel out, ie $p_{tot} = cst.$ over solutions. Consequently:

*The center of mass of a closed system moves with uniform velocity.*

Since the *center of mass* of the system, $q_{cm} := \frac{\sum_j m_j q_j}{\sum_j m_j}$, has $\ddot{q}_{cm} = 0$. Hence:

*One may always study a particular motion of a closed system in an inertial reference frame for which the center of mass is fixed at the origin.*

We have made use of another 'law of motion' in the above derivation, known as the:

**Principle of superposition:** The result of applying two forces $f_1, f_2$ to a particle is that of applying the force given by their vector sum: $f = f_1 + f_2$.

To establish the next general conservation law, we will make use of another 'law of motion':

**Galilean principle of relativity:** Closed systems obey the same equations of motion in all inertial frames.

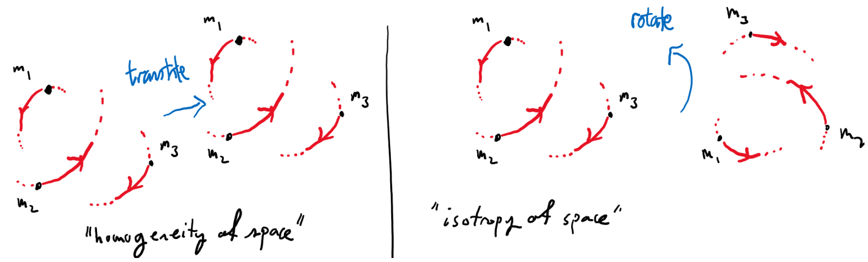


Figure 10. The Galilean principle reflects some intuitive properties that one would expect the 'correct' physical laws governing motion to obey. For instance if an entire closed system is translated by a fixed amount its evolution is 'the same': following its original motion translated by the fixed amount. Likewise for rotating the system by a fixed amount or considering the laws of physics at different times. Additionally, 'Galilean boosts' reflect that if a fixed velocity is added to all the initial velocities of the constituents of a closed system, their resulting motion is the original motion translated with this uniform velocity.

The most general force law arising from Newton's laws, would be a system of the form $\ddot{q} = f(q, \dot{q}, t)$. Galilean relativity imposes strong conditions on the forces in closed systems, namely: time independent, depending only on the mutual positions $(q_j - q_k)$ and mutual velocities $(\dot{q}_j - \dot{q}_k)$, as well as being $\mathrm{O}_3$-equivariant: $f(A(q_j - q_k), A(\dot{q}_j - \dot{q}_k)) = Af(q_j - q_k, \dot{q}_j - \dot{q}_k)$ for any $A \in \mathrm{O}_3$. In particular, we obtain:

*In a closed system with velocity independent forces, the total angular momentum is conserved.*

Indeed, the *total angular momentum* is

$$\vec{C}_{tot} := \sum_j q_j \times p_j.$$

Considering the forces $f_{jk} = -f_{kj}$ on an interacting pair and a reflection, $R$, over the line joining them, we have from the Galilean principle that: $Rf_{jk}(q_j - q_k) = f_{jk}(q_j - q_k)$. Hence $f_{jk} \sim q_j - q_k$ is directed along the line joining the pair. It now follows by differentiation and grouping terms (with N3) that:

$$\dot{\vec{C}}_{tot} = \sum_{j<k} (q_j - q_k) \times f_{jk} = 0.$$

For our last general conserved quantity –the energy– we will need to first consider the physical concept of 'work'. Given a field of forces, $f$, in space and a path, $\gamma \subset \mathbb{R}^3$, the *work* done by the forces when one moves an object along $\gamma$ is:

$$W := \int_\gamma f \cdot d\gamma.$$

The field of forces is called *conservative* if the work depends only on the endpoints of the path. Equivalently, $f = \nabla U$ for some function $U$ called the *force function* of the field of forces, and $V := -U$ is called the *potential energy* of the force field (these functions defined uniquely upto the addition of a constant, by eg, $U(q) := \int_{q_0}^{q} f \cdot d\gamma$ for $q_0$ some fixed point and $\gamma$ a path from $q_0$ to $q$).

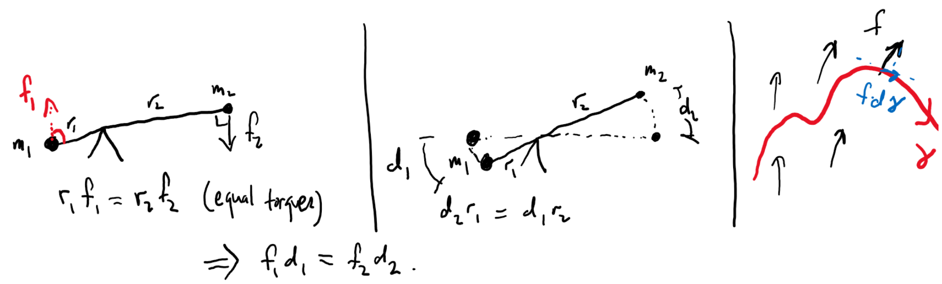


Figure 11. The concept of work may be motivated by considering a *lever*. Exerting a force $f_2$ on one end of the lever produces a force $f_1$ on the other end with $f_1 r_1 = f_2 r_2$ (balance of torques). When one end of the lever moves a distance $d_2$ the other end moves a distance $d_1$, with $d_1 r_2 = d_2 r_2$, so that the products of the forces over the distances displaced during the movement (the *works* done on the ends of the lever) are equal. To move a general particle through a given force field one defines the work done by the force field during the movement as the (integral of the) product of the distance traveled by the component of force along the direction traveled.

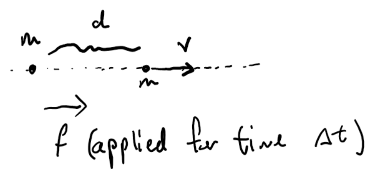


Figure 12. When a (constant) force is applied for a time interval $\Delta t$ to accelerate a particle of mass $m$ from rest to velocity $v$, one has that the force must be applied over a distance $d = \frac{|f|\Delta t^2}{2m}$ and the resulting velocity is $v = \frac{f\Delta t}{m}$. The amount of 'input work' required to achieve such a velocity is thus $|f|d = \frac{m|v|^2}{2}$, called the *kinetic energy* of a particle with mass $m$ and velocity $v$.

The *energy* of a particle subject to a conservative force field is the amount of 'stored work' in its current state (position and velocity), ie the sum of its potential and kinetic energies. It is a conserved quantity.

EXAMPLE:

- For a particle of mass $m$ moving under uniform gravity: $f = (0, 0, -mg)$, the work along a path $\gamma(s) = (x(s), y(s), z(s)), s \in [0, 1]$ with $z(0) = 0, z(1) = z$ is:

$$W(\gamma) = \int_\gamma f \cdot d\gamma = -\int_0^1 mg\frac{dz}{ds}\, ds = -mgz =: U$$

and the potential is $V = mgz$. For a trajectory $q(t) = (x(t), y(t), z(t))$ of the system (satisfying $f = m\ddot{q}$) we have over any time interval $t \in [t_0, t_1]$:

$$U(q(t_1)) - U(q(t_0)) = \int_{q(t_0)}^{q(t_1)} f \cdot dq = \int_{t_0}^{t_1} m\ddot{q} \cdot \dot{q}\, dt = \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{m|\dot{q}|^2}{2}\right)\, dt = \frac{m|\dot{q}(t_1)|^2}{2} - \frac{m|\dot{q}(t_0)|^2}{2}$$

or, rearranging, that the energy:

$$E := \frac{m|\dot{q}|^2}{2} + V(q) = \frac{m|v|^2}{2} + mgz$$

is a conserved quantity over trajectories.

To extend this principle to some general classes of systems of interacting particles, we consider what is the 'stored work' in a given state of the system. The *total kinetic energy* is simply:

$$K := \sum_j \frac{m_j|\dot{q}_j|^2}{2}.$$

As for the total potential energy, we consider some fixed reference configuration of the particles: $q^o = (q_1^o, ...., q_N^o)$ and a path $\gamma(s) = (\gamma_1(s), ..., \gamma_N(s)), s \in [0,1]$ with $\gamma(0) = q^o$ and $\gamma(1) = q = (q_1, ..., q_N)$ a general configuration. Then:

$$W(\gamma) := \sum_j \int_\gamma f_j \cdot d\gamma_j = \int_\gamma f \cdot d\gamma$$

is the *total work* to bring the system from $q^o$ to $q$ along the path $\gamma$, and we say the system is *conservative* if this work depends only on the endpoints $q, q^o$. A *force function* for the system is $U(q) := \int_{q^o}^q f \cdot d\gamma$ where $\gamma$ is a path from $q^o$ to $q$, and a *total potential energy* of the system is $V(q) := -U(q)$. The *total energy*:

$$E := K + V$$

of such systems is a conserved quantity. Moreover,

*A closed system (satisfying the Galilean principle) with velocity independent forces is conservative. In particular, its total energy is a conserved quantity.*

EXAMPLE:

- Consider a system of two particles $q_1, q_2$ of masses $m_1, m_2$ (a *2-body problem*) subject to velocity independent forces from mutual interaction. By the Galilean principle, the equations of motion are:

  $$m_1 \ddot{q}_1 = \frac{f(r)}{r}(q_2 - q_1), \quad m_2 \ddot{q}_2 = \frac{f(r)}{r}(q_1 - q_2)$$

  where $q := q_2 - q_1, r := |q|$ and $f : \mathbb{R}_+ \to \mathbb{R}$ gives the strength of the force (positive for attraction and negative for repulsion).

  By the above, the momenta: $p = p_1 + p_2, \vec{C} = q_1 \times p_1 + q_2 \times p_2$ (with $p_j = m_j \dot{q}_j$) are conserved. The system is conservative, as for $(q_1^o, q_2^o)$ a reference configuration, the integral:

  $$\int_{(q_1^o, q_2^o)}^{(q_1, q_2)} f_{21} \cdot dq_1 + f_{12} \cdot dq_2 = -\int_{q_o}^{q} \frac{f(r)}{r} q \cdot dq = -\int_{r_o}^{r} f(r)\ dr =: U(r)$$

  is path independent, with $V(r) = -U(r)$ a potential energy so that $E = \frac{m_1|\dot{q}_1|^2 + m_2|\dot{q}_2|^2}{2} + V(r)$ is conserved. The particles relative position vector $q := q_2 - q_1$ satisfies:

  $$\mu \ddot{q} = -\frac{f(r)}{r} q$$

  where $\mu := \frac{m_1 m_2}{m_1 + m_2}$. Consider a center of mass zero inertial frame: $m_1 q_1 + m_2 q_2 = 0 = p_1 + p_2$. Then:

  $$q_1 = -\frac{m_2}{m_1 + m_2} q, \quad q_2 = \frac{m_1}{m_1 + m_2} q$$

  So that describing the motion of $q$ completely determines the motions of $q_j$. The integrals are now:

  $$\vec{C} = q \times \mu \dot{q}, \quad E = \frac{\mu |\dot{q}|^2}{2} + V(r).$$

  Hence the motion takes place in a fixed plane $(\vec{C})^\perp$, and in polar coordinates on this plane, we have:

  $$C = \mu r^2 \dot{\theta}, \quad E = \frac{\mu \dot{r}^2}{2} + \frac{C^2}{2\mu r^2} + V(r)$$

  and the motion may be described by the method of 'effective potential wells' (see Arnold's §4, 8 or the following section of our notes).

Exercises:

1. Let $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ be coordinates from an inertial frame. If $(\tilde{x},\tilde{t}) \in \mathbb{R}^3 \times \mathbb{R}$ are a system of coordinates from another inertial frame, show they are related by a change of variable[1] of the form: $(\tilde{x},\tilde{t}) = (Ax + b + tc, t + t_o)$ for $A \in \mathrm{O}_3, b, c \in \mathbb{R}^3, t_o \in \mathbb{R}$.

2. Suppose a system of particles $q_1, ..., q_N$ of particles of masses $m_1, ..., m_N$ follow the equations of motion $m_j \ddot{q}_j = f_j(q_k, \dot{q}_k, t)$ in an inertial frame and satisfy the Galilean principle of relativity. Show the forces are of the form $f_j(q_k - q_l, \dot{q}_k - \dot{q}_l)$ and are $\mathrm{O}_3$-equivariant: $f_j(A(q_k - q_l), A(\dot{q}_k - \dot{q}_l)) = Af_j(q_k - q_l, \dot{q}_k - \dot{q}_l)$ for $A \in \mathrm{O}_3$.

3. Let $q_1, q_2$ be particles of mass $m_1, m_2$ forming a closed system. Show that if the forces $f_{12} = m_2\ddot{q}_2 = -f_{21} = -m_1\ddot{q}_1$ only depend on the positions $q_1, q_2$ then they have the form:
$$f_{kj} = \varphi(r)(q_j - q_k)$$
where $r = |q_j - q_k|$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is some function.

4. For $\gamma$ a path in $\mathbb{R}^3$, show that the line integral $\int_\gamma \gamma \cdot d\gamma$ depends only on the endpoints of $\gamma$ (given by $\int_\gamma d(\frac{r^2}{2}) = \int_\gamma r dr$ where $r$ is the distance from the origin).

5. Consider a closed system of particles $q_1, ..., q_N$ with velocity independent forces. Show that the total energy is conserved.

6. Consider a pair of particles of masses $m_1, m_2$ moving freely towards eachother along a line with initial velocities $v_1, v_2$. Suppose when they collide that they subject eachother to *constant* forces $f_{21}, f_{12}$ (directed along the line) for an interval of time $\Delta t$. Let $\tilde{v}_1, \tilde{v}_2$ be the particles new velocities after this interaction. Show that $m_1 v_1 + m_2 v_2 = m_1 \tilde{v}_1 + m_2 \tilde{v}_2 \iff f_{21} = -f_{12}$.

7. Consider a lever with masses $m_1, m_2$ at its ends at distance $r_1, r_2$ from its pivot.

   (a) Show that if a force $f_1$ is applied perpendicularly to the lever at $m_1$ then it results in a force $f_2$ perpendicular to the lever at $m_2$ with: $|f_1|r_1 = |f_2|r_2$.

   (b) Show that if the lever with the masses at its ends is in a uniform gravitational force field, then it will balance when $m_1 r_1 = m_2 r_2$.

8. Suppose a constant force $f$ is applied to a particle of mass $m$ initially at rest to accelerate the mass to a velocity $v$. Show that $\frac{m|v|^2}{2}$ is the work done by the force during this process.

9. (a) Consider particles $q_1, ..., q_N$ of masses $m_1, ..., m_N$. Given a partition $I_1, ..., I_k$ of $\{1, ..., N\}$ let $Q_j$ be the center of mass of the particles indexed by $I_j$ and $M_j$ the total mass of the particles indexed by $I_j$. Show that
$$q_{cm} = Q_{cm} = \frac{M_1 Q_1 + ... + M_k Q_k}{M_1 + ... + M_k}.$$

   (b) Show that the medians of a triangle are concurrent.

   (c) Show Ceva's theorem: for $A, B, C$ vertices of a triangle with $A', B', C'$ points on the segments $BC, CA, AB$ respectively then the lines $AA', BB', CC'$ are concurrent if and only if $\frac{|AB'|}{|B'C|}\frac{|CA'|}{|A'B|}\frac{|BC'|}{|C'A|} = 1$.

[1] Such transformations forming what is called the *Galilean group*.

## §4 central forces, the inverse square law

We continue our study of 2-body problems, in particular some particular properties motivating Newton's 'inverse square law'. In the last section, we reduced a two-body problem to the uniform motion of its center of mass and a *central force problem*:

$$\mu\ddot{q} = -\frac{f(r)}{r}q, \quad q = q_2 - q_1 \in \mathbb{R}^3, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Example:

- We consider $f(r) = \mu r$, known as the *harmonic oscillator* or *Hooke law*. The general solution is $q(t) = q_o \cos t + v_o \sin t$ and the trajectories move along ellipses centered at the origin. In this case, there are an abundance of conserved quantities. Since angular momentum is constant, each motion takes place in a fixed plane, so we may as well consider the planar case, identified with the complex numbers, with positions and velocities: $(q, v) \in \mathbb{C}^2$. Then we have first integrals:

$$e = |v|^2 + |q|^2, \quad c = 2iq \cdot v, \quad g = Re(v^2 + q^2), \quad h = Im(v^2 + q^2).$$

  There is interesting geometry appearing in this seemingly simple problem when one considers describing the (invariant) level sets in $\mathbb{C}^2$ obtained by fixing various values of these integrals. Of main interest are the energy ($\sim e$) and momentum ($\sim c$) integrals. Writing:

$$e + c = |v + iq|^2, \quad e - c = |v - iq|^2$$

  we see that for $|c| < e$ the energy-momentum surface is topologically a torus, for $|c| = e$ a circle, and empty for $|c| > e$.[1] To see the geometry associated to fixing other integrals, one may take $z = q + iv, w = \bar{q} + i\bar{v}$, so that the integrals are written as:

$$2e = |z|^2 + |w|^2, \quad 2c = |z|^2 - |w|^2, g = Re(z\bar{w}), \quad h = Im(z\bar{w})$$

  and one finds $\mathbb{C}^2 \backslash 0 \to S^2 \subset \mathbb{R}^3, (q, \dot{q}) \mapsto \frac{(g,h,c)}{e}$ gives the *Hopf fibration* (see exercises).

The appearance of the additional first integrals to energy and momentum in the Hooke problem, is a rather exceptional situation. However, the geometry of invariant tori associated to the energy-momentum integrals is a somewhat less exceptional situation. During the course we will encounter this geometry in various situations and eventually explain that it is typical to systems having 'enough' first integrals.

Example:

- We consider certain central forces defined by *power laws*: $f(r) = \frac{\mu}{r^A}$. We have first integrals:

$$\vec{c} := \vec{C}/\mu = q \times \dot{q}, \quad e := E/\mu = \frac{|\dot{q}|^2}{2} + V_A(r)$$

  where $V_A(r) = \begin{cases} \frac{1}{(1-A)r^{A-1}} & A \neq 1 \\ \log r & A = 1 \end{cases}$, by choosing the seperation, $r_o$, of a reference configuration appropriately ($r_o = 0$ for $A < 1$, $r_o = 1$ for $A = 1$ and $r_o \to \infty$ for $A > 1$). We consider here the case $3 > A > 1$ and set $\alpha := A - 1 \in (0, 2)$. Fixing the direction of $\vec{c}$, the solutions all lie in a fixed plane, so in polar coordinates on this plane the integrals are:

$$c = r^2\dot{\theta}, \quad e = \frac{\dot{r}^2}{2} + \frac{c^2}{2r^2} - \frac{1}{\alpha r^\alpha}.$$

  The angular momentum integral has a pleasing geometrical meaning. Its norm is proportional to the rate at which the sectorial area from the origin is swept along trajectories.

[1] By varying the angular momentum and keeping the energy fixed, one obtains a decomposition of the 3-sphere ($e = cst.$) as two solid tori ($D^2 \times S^1$) glued along their boundary.

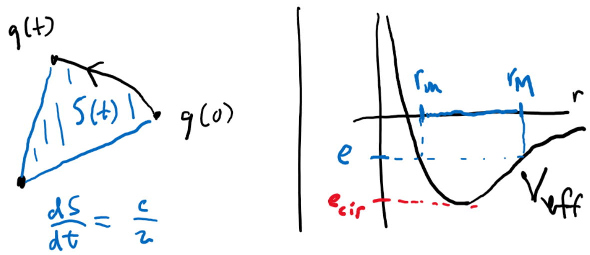


Figure 13. The sectorial area from the origin, $S(t) = \int_{t_o}^{t} \frac{1}{2} r^2 \dot\theta \; dt$, is swept out at the rate $c/2$. Fixing a non-zero momentum and energy in $[e_{cir}, 0)$ one finds from $e \geq V_{eff}(r)$ that the values of $r$ are bounded, oscillating between $r_m$ and $r_M$.

We now use these integrals to describe the bounded orbits with fixed values of $c \neq 0$ and $e$. To see the effect of $e = cst.$, we have $e \geq \frac{c^2}{2r^2} - \frac{1}{\alpha r^\alpha}$ with equality only when $\dot r = 0$. The values of $r$ are restricted by considering the graph of the *effective potential*: $V_{eff}(r; c) := \frac{c^2}{2r^2} - \frac{1}{\alpha r^\alpha}$. Since $\alpha \in (0, 2)$, we find that bounded motions occur only for certain negative values of the energy, where they are constrained to lie in an anulus: with $r(t)$ oscillating between $r_m$ and $r_M$, the minimal and maximal values of $r$ determined by solving $e = V_{eff}(r; c)$.

To describe the motion in this annulus, we recall that the sectorial area is swept at a constant rate along the motions. Hence as the particle oscillates between its extremal values its angle changes by a fixed amount: $\Phi(e, c)$. This angle may be expressed explicitely as the indefinite integral:

$$\Phi = \int_{r_m}^{r_M} d\theta = \int_{r_m}^{r_M} \frac{c \; dr}{r^2 \sqrt{2(e - V_{eff}(r; c))}}.$$

In general, the value of this increment angle $\Phi$ has a non-trivial dependence on $e, c$. Depending upon whether it is a rational multiple of $\pi$ determines whether the orbits are all periodic or densely fill the annulus. Among the inverse power laws there are two exceptional cases to this picture, namely the Hooke law, where one has $\Phi(e, c) \equiv \frac{\pi}{2}$, and the inverse square law, $A = 2$, where one has $\Phi(e, c) \equiv \pi$ (for negative energies). In fact, *Bertrand's theorem* states that among *all* central force laws, the only ones for which all bounded orbits are closed (periodic) are precisely the inverse square law and Hooke.

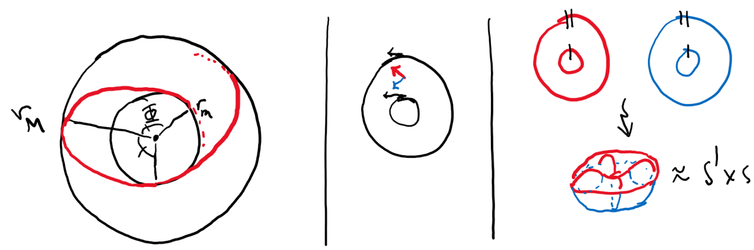


Figure 14. For $c \neq 0$ and $e$ leading to bounded orbits, the motion in the annuli $r_m \leq r \leq r_M$ is determined by the angle $\Phi$, between when an orbit reaches these extremal values. In the position-velocity space, the bounded orbits with fixed energy and momentum form an invariant torus: at interior points to the annulus there are two possible velocities compatible with the energy-momentum condition, while on the boundary just one.

Some of the most impressive predictions of classical mechanics have come in astronomy –for instance the discovery of Neptune– as applications of the following *inverse square law*[1]:

[1]The books: H. Pollard, *Celestial mechanics.* Vol. 18. American Mathematical Soc., (1976). V. Szebeheley, *Adventures in Celestial Mechanics: A First Course in the Theory of Orbits*, Univ. of Texas Press; 1st edition (1989). (the first edition is much better). V. Szebehely, *Theory of orbits: The restricted problem of three Bodies.* Elsevier, (2012). are good references. One can learn much as well from the researcher's sites at IMCCE, eg the articles/lectures of Albouy, Chenciner, Féjoz.

**Newton's universal law of gravitation:** the force due to gravity between two particles is proportional to the product of their masses over the distance squared between them: $|f_{grav}| = G\frac{m_1 m_2}{r_{12}^2}$.

Here $G \approx 6.674 \times 10^{-11} \frac{m^3}{kg \cdot sec^2}$ is the *universal gravitational constant*, whose value depends only on the units used. Note that by the principle of superposition, the inverse square law determines gravitational forces for systems of more than two particles.

The main experimental results motivating the inverse square law are:

**Kepler's laws of planetary motion (K1-3):**

1. objects orbit the sun along conics, having a focus at the sun.
2. The sectorial area swept by the object increases at a constant rate.
3. The period of an elliptic orbit squared is proportional to the cube of its major axis.

The rate in K2 depends on the orbit of the object, while the proportionality constant in K3 is the same for any object. Kepler deduced these laws from careful study of Brahe's observations. We have:

*The trajectories of a force field satisfy Kepler's laws* $\iff$ *the force is an inverse square law.*

That is: solutions to $\ddot{q} = F(q)$ satisfy K1-K3 iff $F(q) = -k\frac{q}{|q|^3}$ for some constant $k > 0$. To establish this (taking the sun as the origin) one may reason as follows:

Since the orbits are contained in planes, $q \times \dot{q}$ has a fixed direction over an orbit. Its magnitude is the rate of sectorial area increase, hence $q \times \dot{q}$ is constant along trajectories. Differentiating gives:

$$0 = q \times F(q) \Rightarrow F(q) = -\frac{f(q)}{|q|}q$$

for some $f : \mathbb{R}^3 \to \mathbb{R}$, ie we are in a central force field. Next, we take polar coordinates, $(r, \theta)$, in an orbital plane in which the orbit is given by:

$$r = \frac{p}{1 + e\cos\theta} \qquad \text{(equation of a conic with focus at the origin in polar coordinates)}$$

while the equations of motion are (with $C/2$ the constant rate of sectorial area increase):

$$\ddot{r} = \frac{C^2}{r^3} - f(q), \quad r^2\dot{\theta} = C.$$

There is a convenient change of variable: $\rho = 1/r$, for which a conic with focus at the origin satisfies:

$$\rho'' + \rho = \frac{1}{p}$$

using $' = \frac{d}{d\theta}$. Using chain rule and $r^2 d\theta = Cdt$, we have: $\dot{r} = -\frac{\dot{\rho}}{\rho^2} = -C\rho' \Rightarrow \ddot{r} = -\frac{C^2}{r^2}\rho''$. Substitution into the equations of motion gives:

$$-C^2\rho^2\rho'' = C^2\rho^3 - f(q) \Rightarrow f(q) = \frac{C^2}{p}\rho^2 = \frac{C^2}{p}\frac{1}{r^2}.$$

Now let $a, b$ be the semi-major/minor axes of an elliptical orbit. Then $p = b^2/a$ and its period is $T = \frac{2\pi ab}{C}$,

$$\Rightarrow \frac{C^2}{p} = 4\pi^2\frac{a^3}{T^2} = k$$

or, in summary, $F(q) = -k\frac{q}{|q|^3}$ as claimed.

Kepler's laws are a limiting case of Newton's inverse square law applied to two-body problems. Namely, consider the two-body problem for masses $m_1, m_2$ under Newtonian gravity and suppose that $m_1 >> m_2$, say $m_2 = \varepsilon m_1$ for $\varepsilon$ small. In a center of mass zero frame:

$$\ddot{q} = -G(m_1+m_2)\frac{q}{|q|^3} = -G(1+\varepsilon)m_1\frac{q}{|q|^3}, \quad q_1 = -\frac{\varepsilon}{1+\varepsilon}q, \quad q_2 = \frac{1}{1+\varepsilon}q$$

As $\varepsilon \to 0$ (as $m_1 = M_S$ the mass of the 'sun' grows much larger than that of the 'planet', $m_2$) we have:

$$\ddot{q} = -GM_S\frac{q}{|q|^3}, \quad q_1 = q_S = 0, \quad q_2 = q.$$

Thus K3 may be made more precise by: $4\pi^2 a^3 = G(M_S + m_p)T^2$, so that the major-axis and period relation depends on the mass, $m_p$, of the planet as well. In this discussion applied to the solar system, we have only taken account of forces between the sun and *a* planet, a good model, because we were able to solve it. More realistically, the planets all subject eachother to mutual attractions, a much more complicated problem.

We will mention another particular property of the inverse square law regarding its corresponding *potential theory*. A large number of particles occupying a region in space may be imagined as a *continuum*: a region $\Omega \subset \mathbb{R}^3$, equipped with a *mass density*: $\rho : \mathbb{R}^3 \to \mathbb{R}_+$ (with $\rho = 0$ on $\Omega^c$). The principle of superposition implies that the gravitational field produced by such a continuum has potential:

$$V_\Omega(q) = -\int_\Omega \frac{\rho(x)\ d^3x}{|q-x|}.$$

Such continua are said to be *homogeneous* when their densities are constant functions on $\Omega$. Now, it can be shown that the inverse square law is the only force law with the following property:

*The force field produced by a homogeneous spherical shell vanishes in the shells interior, while in its exterior is the same as that of a particle at its center with its total mass.*

In particular, it follows that the force field produced by a homogeneous solid ball in its exterior is the same as that of a particle at the ball's center with the ball's total mass.

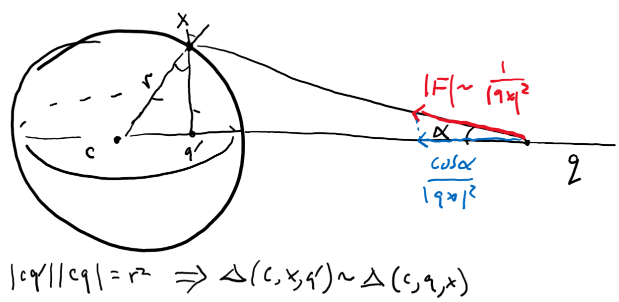


Figure 15. The gravitational force due to a homogeneous spherical shell of mass $M$ may be calculated directly. Alternately, for exterior points, $q$, one may consider the geometry of the spherical inversion, $q'$ of $q$. One finds that for any $x$ on the shell, the triangles $\Delta(c,x,q')$ and $\Delta(c,q,x)$ are similar. It follows that the total force on $q$ due to the shell has strength as: $\int_{x\in S^2_r}\frac{\sigma \cos\alpha dA}{|xq|^2} = \int_{x\in S^2_r}\frac{\sigma|xq'|^2 d\Omega_{q'}}{|xq|^2} = \int_{x\in S^2_r}\frac{\sigma r^2 d\Omega_{q'}}{|cq|^2} = \frac{M}{|cq|^2}$ where $\sigma$ is the mass density on the sphere and $d\Omega_{q'}$ the element of solid angle from $q'$.

These properties may be encapsulated in the potential $V_\Omega$ due to a mass density $\rho$ satisfying the *Poisson equation:*[1]

$$\Delta V_\Omega = \rho.$$

[1]Here $\Delta$ is the *Laplacian* operator. See, eg, the article: T. Tokieda, *Tides: A Tutorial.* Tides in Astronomy and Astrophysics. Springer, Berlin, Heidelberg, 1-30, (2013). (online here).

Exercises:

1. Consider planar motions under Hooke's law, $\ddot{q} = -q$, with $q \in \mathbb{C}$ and $v := \dot{q} \in \mathbb{C}$.

   (a) For $z = q + iv, w = \bar{q} + i\bar{v}$, show that $z/w \in \mathbb{C}$ (when $w \neq 0$) stays constant over trajectories.

   A complex line in $\mathbb{C}^2$ is a subspace $\{\lambda(z,w) : \lambda \in \mathbb{C}\}$ with $(z,w) \neq (0,0)$. The complex projective line, $\mathbb{CP}^1$, is the set of complex lines in $\mathbb{C}^2$. We write $(z : w) \in \mathbb{CP}^1$ for the complex line spanned by $(z,w) \in \mathbb{C}^2 \backslash (0,0)$.

   (b) The *Hopf map* is $h : \mathbb{C}^2 \backslash (0,0) \to \mathbb{CP}^1, (z,w) \mapsto (z : w)$. It may be represented in coordinates as follows. Let $a : \mathbb{CP}^1 \backslash \{w \neq 0\} \to \mathbb{C}, (z : w) \mapsto z/w \in \mathbb{C}$ and $s : \mathbb{C} \to S^2 \backslash \{(0,0,1)\}, u \mapsto (\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1})$. Show $s \circ a$ extends to a map $f : \mathbb{CP}^1 \to S^2$, and give an explicit formula for $H := f \circ h : \mathbb{C}^2 \backslash (0,0) \to S^2$.[1]

2. For an inverse power force law $f(r) = \mu/r^A$, and a fixed non-zero value of momentum, determine for which values of energy it is possible to have bounded orbits.

3. (a) For $a < b$, show that $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi$.

   Consider the following central force laws with a fixed non-zero momentum value: $c > 0$.

   (b) Show that for the Hooke law, we have $\Phi = \pi/2$ (suggestion: consider the substitution $u = 1/r^2$).

   (c) Show that for an inverse square law and a bounded (negative energy) orbit we have $\Phi = \pi$ (suggestion: consider the substitution $u = 1/r$).

   (d) For the inverse square law and non-negative energies, determine $\Phi := \int_{r_m}^{\infty} d\theta$.

4. Consider a homogeneous distribution of mass over the surface of a sphere. Show that:

   (a) a point inside the sphere experiences no gravitational force

   (b) a point outside the sphere experiences the same force as that due to a particle at the center of the sphere with the total mass of the sphere.

5. Consider a homogeneous distribution of mass over a solid ball. Show that:

   (a) a point outside the ball experiences the same force as that due to a particle at the center of the ball with the total mass of the ball.

   (b) a point inside the sphere experiences a force directed towards the center of the ball with a strength proportional to its distance from the center of the ball (a Hooke law).

6. Consider the system $\ddot{x} = \frac{dU}{dx}(x)$ for $x \in \mathbb{R}$. Suppose the system admits periodic orbits (solutions tracing closed curves in the $(x, \dot{x})$-plane) for some energy values. Let $A(E)$ be the area in the $(x, \dot{x})$-plane enclosed by such a periodic orbit of energy $E$ and $T(E)$ the period of the orbit. Show that $\frac{dA}{dE} = T$.

[1] One may find some beautiful pictures by searching 'Hopf fibration'. These pictures consider a sterographic projection of $S^3 \subset \mathbb{C}^2$ to $\mathbb{R}^3$ and draw the circles which are pre-images of points on $S^2$ under $H$ (they are all linked).

**§5 constraints**

We now consider how to treat certain systems subject to *constraints*, namely[1] when the positions $q = (q_1, ..., q_N) \in \mathbb{R}^{3N}$ of a system of particles are only permitted to take certain values:

$$q \in \Sigma \subset \mathbb{R}^{3N}.$$

Moreover, this constraint set $\Sigma$ is assumed to be defined (locally) by a system of (smooth) equations, eg:

$$\Sigma = \{c_1(q) = 0, ..., c_k(q) = 0\}$$

with $c_j : \mathbb{R}^{3N} \to \mathbb{R}$ the constraint conditions.

EXAMPLES:

- The bob of a (planar) pendulum, $q = (x, y, z)$, is subject to the constraint that it lies in a fixed circle, eg $|q| = \ell, z = 0$.
- The bob of a spherical pendulum is subject to the constraint that it lies in a fixed sphere, eg $|q| = \ell$. More generally, one might consider a particle constrained to lie in some fixed surface $\Sigma \subset \mathbb{R}^3$.
- The particles, $q_1, ..., q_N$, constituting a *rigid body*[2] are subject to the constraint that their mutual distances and angles remain fixed: $(q_j - q_k) \cdot (q_\ell - q_m) = cst.$.

The fundamental question to address is: given a system subject to constraints and forces, $f_j$ applied to the particle $q_j$ of the system, what system of ode's will govern the resulting motions? First:

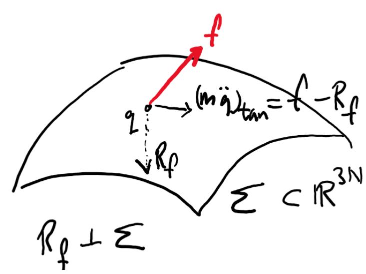


Figure 16. One may visualize that applying forces, $f$, to a configuration $q \in \Sigma$ produces a *reaction force*, $R_f$, 'absorbing' some of the applied forces $f$, and leading to a net force $f - R_f$ acting on the particle to determine its tangential acceleration. d'Alembert's principle posits that these reaction forces to a given applied force are the component of the force normal to the constraint set.

**Principle of static equilibrium:** a system of particles $(q_1, ..., q_N) \in \Sigma \subset \mathbb{R}^{3N}$ subject to forces $f_j$ is in *equilibrium* when

$$0 = \sum f_j \cdot \delta q_j = f \cdot \delta q$$

for any *virtual velocity* $\delta q = (\delta q_1, ..., \delta q_N) := \frac{d}{ds}|_{s=0}\gamma(s)$ where $\gamma(s) \in \Sigma$ and $\gamma(0) = (q_1, ..., q_N)$.

Equivalently, the system is in equilibrium when the forces it is subject to do no work over paths consistent with the constraint. Consequently, one will often encounter it phrased as: the reaction forces due to the constraint do no work and the forces $f_j$ lead to conditions of equilibrium when they are cancelled by the reaction forces. The equations of motion of a system with constraints when subject to general forces may be characterized by the:

[1] A case of what are called *holonomic constraints*.

[2] See eg the article T. Tokieda, *Spinning bodies: a tutorial.* Dynamics of Extended Celestial Bodies and Rings: 1-22, (2006). (online here).

**d'Alembert principle:**[1] the motions, $q_j(t)$, of a system of particles $(q_1, ..., q_N) \in \Sigma \subset \mathbb{R}^{3N}$ subject to the forces $f_j$ satisfy:

$$0 = \sum (f_j - m_j \ddot{q}_j) \cdot \delta q_j = (f - m\ddot{q}) \cdot \delta q$$

for any virtual velocity $\delta q = (\delta q_1, ..., \delta q_N)$.

In different language, the constraint set $\Sigma \subset \mathbb{R}^{3N}$ is a submanifold and a virtual velocity is a tangent vector to this submanifold: $\delta q \in T_q\Sigma$. The condition for the forces $f = (f_1, ..., f_N)$ to admit equilibrium is that it be perpendicular to $\Sigma$, and in general the equations of motion are equality among the tangential components, $f_{tan} = (m\ddot{q})_{tan}$, to $\Sigma$.

To benefit from the d'Alembert principle, one may take advantage of its characterization being independent of coordinates. If one can write the forces, accelerations and virtual velocities in some coordinate system, one may then apply the principle in these coordinates to obtain the equations of motion.

Examples:

- The particle on the pendulum may be given by $q = \ell e^{i\theta} \in \mathbb{C}$ with $f = mg \in \mathbb{R} \subset \mathbb{C}$. Then $\delta q = \ell i e^{i\theta}\delta\theta$ for $\delta\theta \in \mathbb{R}$ and $\ddot{q} = -\ell\dot{\theta}^2 e^{i\theta} + \ell\ddot{\theta} i e^{i\theta}$. d'Alembert's principle gives:

$$0 = m\ell(g - \ell\ddot{\theta} i e^{i\theta}) \cdot (ie^{i\theta})\delta\theta, \quad \forall \delta\theta \in \mathbb{R}$$

$$\Rightarrow 0 = (-g\sin\theta - \ell\ddot{\theta})\delta\theta, \quad \forall\delta\theta \in \mathbb{R} \Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

- The particle at the bob of the spherical pendulum may be given by $q = \ell(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $f = mg\hat{k} = (0, 0, mg)$. Then $\delta q = \delta\varphi e_\varphi + \delta\theta\sin\varphi e_\theta$ for $\delta\varphi, \delta\theta \in \mathbb{R}$ and $e_\varphi := (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$, $e_\theta := (-\sin\theta, \cos\theta, 0)$ and $\ddot{q} = \ell(\ddot{\varphi}e_\varphi + \frac{d}{dt}(\dot{\theta}\sin\varphi)e_\theta + \dot{\varphi}\dot{e}_\varphi + \dot{\theta}\sin\varphi\dot{e}_\theta)$. Note that $e_\varphi \cdot e_\varphi = 1, e_\varphi \cdot e_\theta = 0, e_\theta \cdot e_\theta = 1$, hence $e_\varphi \cdot \dot{e}_\varphi = 0, e_\theta \cdot \dot{e}_\theta = 0, \dot{e}_\varphi \cdot e_\theta = -e_\varphi \cdot \dot{e}_\theta = \dot{\theta}\cos\varphi$. Now, we are ready to apply d'Alembert:

$$0 = (g\hat{k} - \ddot{q}) \cdot \delta q = \delta\varphi\left(-g\sin\varphi - \ell\ddot{\varphi} + \ell\dot{\theta}^2\sin\varphi\cos\varphi\right) + \ell\delta\theta\left(\ddot{\theta}\sin^2\varphi + 2\dot{\theta}\dot{\varphi}\sin\varphi\cos\varphi\right), \quad \forall\delta\varphi, \delta\theta \in \mathbb{R}$$

$$\Rightarrow \frac{d}{dt}(\dot{\theta}\sin^2\varphi) = 0, \quad c := \dot{\theta}\sin^2\varphi, \quad \ddot{\varphi} = -\frac{g}{\ell}\sin\varphi + \frac{c^2\cos\varphi}{\sin^3\varphi}.$$

- A *geodesic* on a surface $\Sigma \subset \mathbb{R}^3$ is the trajectory of a free particle (subject to no external forces) constrained to the surface. By d'Alembert's principle, the motion of a geodesic on a surface is determined by the condition that the tangential component of its acceleration is zero. The speed of a geodesic, $|\dot{q}|$, is always a conserved quantity. In general, the second order ode's determining geodesics on a surface are complicated.

  We consider the case of *surfaces of revolution:* revolve the (arc-length) parametrized curve $(x(s), 0, z(s))$ about the $z$-axis. The position of a particle on this surface of revolution may be given by: $q = (x(s)\cos\theta, x(s)\sin\theta, z(s))$. In this case, that the acceleration $\ddot{q}$ is entirely normal to the surface means that it is contained in the $q, \hat{k} = (0, 0, 1)$ plane, in particular $q \times \ddot{q}$ is perpendicular to $\hat{k}$. It follows that the component of angular momentum along the axis of rotation: $(q \times \dot{q}) \cdot \hat{k}$ is conserved, since its time derivative is $(q \times \ddot{q}) \cdot \hat{k} = 0$.

  We may also apply d'Alembert to obtain the explicit equations of motion: $\delta q = \delta s e_s + x\delta\theta e_\theta$ with $e_s = (x'\cos\theta, x'\sin\theta, z'), e_\theta = (-\sin\theta, \cos\theta, 0)$ orthonormal vectors such that $\dot{e}_s \cdot e_\theta = -e_s \cdot \dot{e}_\theta = x'(s)\dot{\theta}$. Also $\ddot{q} = \ddot{s}e_s + \frac{d}{dt}(x\dot{\theta})e_\theta + \dot{s}\dot{e}_s + x\dot{\theta}\dot{e}_\theta$. Hence:

$$0 = \ddot{q} \cdot \delta q = \delta s\left(\ddot{s} - xx'\dot{\theta}^2\right) + \delta\theta\left(2xx'\dot{s}\dot{\theta} + x^2\ddot{\theta}\right), \quad \forall\delta s, \delta\theta \in \mathbb{R}$$

[1] The lectures: A. Albouy, *Some classical integrable problems*, Lectures in Hanoi (2007). Available online here contain further examples and discussion.

$$\Rightarrow c = x^2\dot{\theta}, \quad \ddot{s} = c^2\frac{x'(s)}{x(s)^3}.$$

Observe as well that the $\hat{k}$-angular momentum component, $c$, has a nice geometric interpretation, useful to sketch the geodesics. Namely, for unit speed geodesics, we have $\cos\alpha = \dot{q}\cdot e_\theta = x\dot{\theta} = c/x$ where $\alpha$ is the angle between the velocity $\dot{q}$ of the geodesic and the latitude or horizontal direction $e_\theta$. Hence $c = x(s)\cos\alpha$ (where $x$ is the distance of the particle to the axis of rotation) is constant. In particular geodesics with fixed $c \neq 0$ cannot get too close to the axis of rotation: satisfying $x \geq |c|$. As they get further from the axis they get 'more vertical' ie $\alpha \to \pm\pi/2$, while as they get closer to the axis 'more horizontal' ie $\alpha \to 0, \pi$. In this geometric context, $c$ is often called the *Clairaut integral* for geodesics on surfaces of revolution.

As a final application of d'Alembert's principle, we will derive the equations of motion for a *free rigid body.* There is essentially no difference in considering the case when the rigid body is thought of as a continuum: a solid region with a mass density, versus as a collection of particles. The differences amounting to replacing mass density weighted integrals with mass weighted sums.

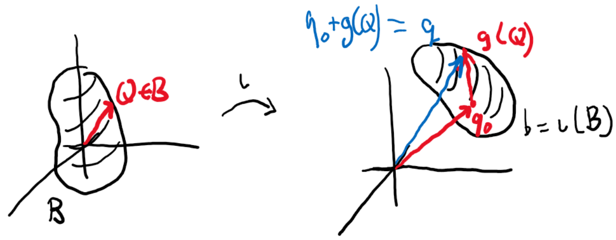


Figure 17. All possible configurations, $b$, of the rigid body are obtained by applying an isometry of space to a *fixed* reference configuration $B$: $b = \iota(B)$. Letting $q_o = \iota(0) \in \mathbb{R}^3$, then the points $Q \in B$ correspond to $q = q_o + g(Q) \in b$ for $g \in \mathrm{SO}_3$ a rotation.

By considering a fixed 'reference' configuration, all configurations of the rigid body are parametrized by $(q_o, g) \in \mathbb{R}^3 \times \mathrm{SO}_3$. We are now in position to apply d'Alembert's principle. The motions of the rigid body are certain curves $(q_o(t), g(t)) \in \mathbb{R}^3 \times \mathrm{SO}_3$. A general virtual velocity is then the field:

$$\delta q = \delta q_o + \delta\vec{\omega} \times (q - q_o), \quad \delta q_o, \delta\vec{\omega} \in \mathbb{R}^3.$$

We will obtain the equations of motion by unraveling the condition $0 = \int_{q\in b} \ddot{q}\cdot\delta q \; d\mu, \ \forall \delta q$, where $d\mu := \rho d^3q$. This may be done in two stages, first we consider $\delta q$ with $\delta\vec{\omega} = 0$ and $\delta q_o \in \mathbb{R}^3$ arbitrary. Then:

$$0 = \int_b \ddot{q}\cdot\delta q_o \; d\mu = \left(\int_b \ddot{q}\; d\mu\right)\cdot\delta q_o, \quad \forall \delta q_o \in \mathbb{R}^3$$

$$\Rightarrow \int_b \ddot{q}\; d\mu = 0.$$

Since the center of mass of the configuration $b$ is $q_{cm} = \frac{\int_b q\; d\mu}{M}, \quad M := \int_b d\mu$, this last condition reads:

$$\ddot{q}_{cm} = M\int_b \ddot{q}\; d\mu = 0$$

so that for a free rigid body, linear momentum is conserved and its center of mass may be taken as the origin of an inertial frame. Moreover, we choose our reference configuration $B$ to have its center of mass at the origin, so that $q_o = q_{cm} \equiv 0$ in this center of mass zero frame. Now, we consider the equations of motion characterizing the rotational part. We have:

$$q = g(Q), \quad \ddot{q} = \dot{\omega}(q) + \omega^2(q), \quad \delta q = \delta\vec{\omega}\times q, \quad \delta\vec{\omega} \in \mathbb{R}^3$$

where $\omega := \dot{g}g^{-1} \in \mathfrak{so}_3$ is a skew-symmetric matrix. Then:

$$0 = \int_b \ddot{q} \cdot \delta q \; d\mu = \delta\vec{\omega} \cdot \left( \int_b q \times (\dot{\omega}(q) + \omega^2(q)) \; d\mu \right), \quad \forall \delta\vec{\omega} \in \mathbb{R}^3$$

$$\Rightarrow 0 = \int_b q \times \dot{\omega}(q) + q \times \omega^2(q) \; d\mu.$$

Technically, this last equation is the equation of motion, since it relates $\ddot{g}$ to $g, \dot{g}$ via $\omega = \dot{g}g^{-1}$. So we could say we are done. However the equation of motion is not very useful in this form. We will perform some algebraic manipulations to put it in a more useful form, and in fact see that it is equivalent to the conservation of angular momentum!

Recall we defined the inertia tensor by $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3, \vec{\omega} \mapsto \vec{c} = \int_b q \times (\vec{\omega} \times q) \; d\mu$, sending an infinitesimal rotation $\vec{\omega}$ to its resulting angular momentum vector. There is a useful extension of this inertia tensor, upon using the identification $\mathbb{R}^3 \leftrightarrow \mathfrak{so}_3$, we take:

$$\mathbb{I} : \mathfrak{gl}_3 \to \mathfrak{so}_3, \quad A \mapsto As - sA^t$$

where $\mathfrak{gl}_3$ are general $3 \times 3$ matrices and $s = \int_b qq^t \; d\mu = \int_b \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \; d\mu$ is symmetric. This extension of the inertia tensor comes from considering that $\mathbb{I} : \mathfrak{so}_3 \to \mathbb{R}^3, \omega \mapsto \int_b q \times (\omega q) \; d\mu$ extends to $\mathbb{I} : \mathfrak{gl}_3 \to \mathbb{R}^3, A \mapsto \int_b q \times (Aq) \; d\mu$. Identifying the right $\mathbb{R}^3$ with $\mathfrak{so}_3$ gives the skew map $v \mapsto \int_b (q \times Aq) \times v \; d\mu = \int_b (q \cdot v)Aq - (Aq \cdot v)q \; d\mu = \left( \int_b Aqq^t - qq^tA^t \; d\mu \right) v = (As - sA^t)v$.

Thus, the equations of motion are: $\mathbb{I}\dot{\omega} + \mathbb{I}\omega^2 = 0$, or:

$$(*) \qquad s\dot{\omega} + \dot{\omega}s = s\omega^2 - \omega^2 s.$$

Still, these are not very convenient since the matrix $s$ depends on the current configuration $b$, ie the region of integration depends on $g$. We obtain another version of the equations of motion by 'pulling back' to the (fixed) reference configuration $B$. Namely, since $q = g(Q)$, we have by change of variable that:

$$s = gSg^{-1}, \quad S = \int_B QQ^t \; d\mu$$

where the symmetric matrix $S$ is obtained by integration over the *fixed* reference configuration $B$. Moreover, $\dot{q} = \omega q = \dot{g}Q = gg^{-1}\dot{g}Q = g\Omega Q$, where $\Omega := g^{-1}\dot{g} \in \mathfrak{so}_3$. The vector $\Omega Q$ represents the velocity $\dot{q}$ on the fixed reference configuration $B$. We have $g\Omega = \omega g$ from which it follows that $g\dot{\Omega} = \dot{\omega}g$. Substitution into $(*)$ yields the equations of motion:

$$(**) \qquad S\dot{\Omega} + \dot{\Omega}S = S\Omega^2 - \Omega^2 S.$$

Solving this 1st order ode for $\Omega(t)$ (note $S$ is a fixed symmetric matrix), then describes the infinitesimal axis of rotation $\mathbb{R}^3 \ni \vec{\Omega}(t) \leftrightarrow \Omega(t) \in \mathfrak{so}_3$ as seen in the reference configuration $B$ and the actual motion $g(t)$ could be –in principle– recovered by a further integration of $\dot{g} = g\Omega(t)$. We will be content with describing the trajectories of the infinitesimal rotation axis.

The two equations of motion $(*), (**)$ are equivalent. Furthermore, they are both equivalent to the conservation of angular momentum:

$$\vec{c} \leftrightarrow c = \mathbb{I}\omega = \omega s + s\omega, \qquad \dot{c} = 0$$

or to the following equation of motion for the angular momentum as seen in the fixed reference configuration:

$$\vec{C} = g^{-1}\vec{c} \leftrightarrow C = \mathbb{I}_B\Omega = \Omega S + S\Omega = g^{-1}cg, \quad \dot{C} = [C, \Omega] = C\Omega - \Omega C$$

And one may describe the infinitesimal rotation axes' trajectories using the integrals of motion:

$$|\vec{C}|^2 = |\vec{c}|^2 = \mathbb{I}_B\vec{\Omega} \cdot \mathbb{I}_B\vec{\Omega}, \quad 2K = \mathbb{I}\vec{\omega} \cdot \vec{\omega} = -\frac{1}{2}tr(\omega\mathbb{I}\omega) = -\frac{1}{2}tr(\Omega\mathbb{I}_B\Omega) = \mathbb{I}_B\vec{\Omega} \cdot \vec{\Omega}.$$

Exercises:

1. Consider the *Huygens pendulum*: a point mass is attached to the free end of a string of half the arc-length of a cycloid fixed at its other end to the cusp of the cycloid, and subject to constant 'vertical' acceleration force. Show that the period of oscillation is constant (independent of the initial conditions).

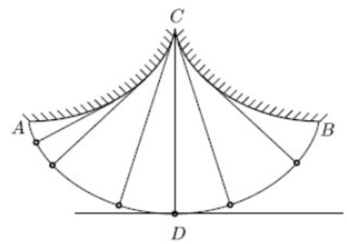


Figure 18. The Huygens pendulum: a point mass oscillates along an involute of the cycloid subject to constant vertical acceleration.

2. (a) Show that a geodesic on a surface $\Sigma \subset \mathbb{R}^3$ has constant speed: $|\dot{q}| = cst$.

   (b) Show that the motions of a point mass constrained to a surface of revolution and subject to a 'vertical' force directed along its axis of rotation admits the component of angular momentum along its axis of rotation as a conserved quantity.

3. Sketch the geodesics on a torus of revolution (revolve about the $z$-axis a circle of radius $r$ in the $xz$-plane with center at distance $R > r$ from the $z$-axis).

4. Sketch the motions of a spherical pendulum.

5. Show that $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$ for $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.

6. (a) For $\Omega_{\vec{\omega}} \in \mathfrak{so}_3$ corresponding to $\vec{\omega} \in \mathbb{R}^3$ and $g \in \mathrm{SO}_3$, show that $g\Omega_{\vec{\omega}}g^{-1} \in \mathfrak{so}_3$ corresponds to $g\vec{\omega} \in \mathbb{R}^3$.

   (b) For $g(t) \in \mathrm{SO}_3$ a curve of rotations and $\omega := \dot{g}g^{-1}, \Omega = g^{-1}\dot{g} \in \mathfrak{so}_3$, show that $g\dot{\Omega} = \dot{\omega}g$.

7. (a) Show that the equations of motion $(**)$ we derived above for the rigid body are equivalent to: $\dot{C} = [C, \Omega]$ which in turn is equivalent to $\dot{c} = 0$.

   (b) Show that the kinetic energy: $K = -\frac{1}{4}tr(\Omega\mathbb{I}_B\Omega) = \frac{1}{2}\mathbb{I}_B\vec{\Omega} \cdot \vec{\Omega}$ is conserved for the free rigid body.

   (c) In a previous exercise, you have shown that $\mathbb{I}_B$ is a symmetric (positive definite) matrix. Consider a rigid body such that the eigenvalues of $\mathbb{I}_B$ are distinct: $0 < I_1 < I_2 < I_3$. Describe trajectories of the free rigid body with $K = \frac{1}{2}$ and various values of $|\vec{C}|^2$ by considering level sets in the space $\mathbb{R}^3 \ni (x, y, z) := (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$.

## §6 variational principle

The trajectories of many mechanical systems may be described variationally, meaning that they are the ‘extremals’ of an ‘action functional’ defined over a certain class of curves.

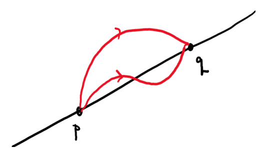


Figure 19. Lines in the plane may be characterized variationally: among all (rectifiable) curves joining two points on the line, the line segment between the points is the curve with the shortest arc-length.

It is worthwhile to consider in some generality how one may set-up and treat variational problems. The set-up consists of a certain *class of curves*, $\Gamma$, along with a *functional*, $A : \Gamma \to \mathbb{R}$ assigning to each curve in the class a number. One seeks curves which are ‘extremals’ –analogues of critical points– of the functional, for instance a *minimizer*: $\gamma_* \in \Gamma$ such that $A(\gamma_*) \leq A(\gamma)$ for all $\gamma \in \Gamma$.

To test whether a given $\gamma \in \Gamma$ is an extremal, one considers *variations* of $\gamma$: a family of curves $\gamma_\varepsilon \in \Gamma$ for $\varepsilon \in (-\delta, \delta) \subset \mathbb{R}$ with $\gamma_0 = \gamma$, and computes: $\frac{d}{d\varepsilon}|_{\varepsilon=0} A(\gamma_\varepsilon)$. For this definition to have sense, one needs to be able to define ‘smooth variations’ and consider ‘smooth functionals’ so the derivative $\mathbb{R} \to \mathbb{R}, \varepsilon \mapsto A(\gamma_\varepsilon)$ is defined. In such a case, an *extremal* is then defined as a curve $\gamma_* \in \Gamma$ such that for any smooth variation, $\gamma_\varepsilon$, of $\gamma_*$ it holds that:

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} A(\gamma_\varepsilon).$$

In our examples, the classes of curves considered will consist of curves in vector spaces, $\gamma : [0, T] \to \mathbb{R}^k$. A variation, $\gamma_\varepsilon$ of such a curve is then said to be smooth when the map $\mathbb{R}^2 \supset (-\delta, \delta) \times [0, T] \to \mathbb{R}^k, (\varepsilon, t) \mapsto \gamma_\varepsilon(t)$ is smooth. The functionals we consider will as well be smooth, namely: for any smooth variation of curves then $\varepsilon \mapsto A(\gamma_\varepsilon)$ is differentiable.

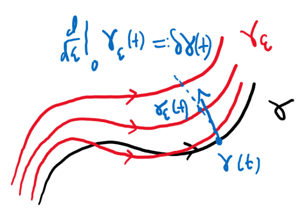


Figure 20. A variation, $\gamma_\varepsilon$, of a curve $\gamma$.

Now, we may state the:

**variational principle:** The trajectories of a system of particles $(q_1, ..., q_N) \in \Sigma \subset \mathbb{R}^{3N}$, subject to conservative forces ($f_j = -\partial_{q_j} V$, for a potential $V = -U : \mathbb{R}^{3N} \to \mathbb{R}$), are exactly the extremals of the functional:

$$q(t) \mapsto \int_{t_0}^{t_1} \sum \left( \frac{m_j |\dot{q}_j(t)|^2}{2} \right) - V(q(t))\ dt =: \int_{t_0}^{t_1} L(q, \dot{q})\ dt$$

over smooth curves $q : [t_0, t_1] \to \Sigma$ with fixed endpoints $q(t_0) = q_0, q(t_1) = q_1$.

*proof:* We may obtain this from the d'Alembert principle (in fact they are equivalent in this setting). Let $q_\varepsilon$ be a variation of a trajectory $q(t)$ of the system with $q_\varepsilon(t_0) = q_0, q_\varepsilon(t_1) = q_1$ fixed. Observe that the vector field $\delta q(t) := \frac{d}{d\varepsilon}|_{\varepsilon=0} q_\varepsilon(t)$ along $q$ consists of 'virtual velocities' (tangent to $\Sigma$) and vanishes at $t_0, t_1$ (fixed endpoints). So d'Alembert: $0 = \sum (f_j - m_j \ddot{q}_j) \cdot \delta q_j(t)$, implies:

$$0 \overset{(*)}{=} \sum \int_{t_0}^{t_1} \partial_{q_j} U \cdot \delta q_j - m_j \ddot{q}_j \cdot \delta q_j \; dt = \sum \int_{t_0}^{t_1} \partial_{q_j} U \cdot \delta q_j + m_j \dot{q}_j \cdot \dot{\delta q}_j \; dt$$

using integration by parts. On the other hand, differentiating under the integral sign, we have:

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \int_{t_0}^{t_1} L(q_\varepsilon, \dot{q}_\varepsilon) \; dt = \sum \int_{t_0}^{t_1} m_j \dot{q}_j \cdot \dot{\delta q}_j + \partial_{q_j} U \cdot \delta q_j \; dt = 0$$

so that trajectories are extremals as stated. Conversely, if $q(t)$ is an extremal, then equation $(*)$ holds over any time interval $t_0, t_1$ which implies the integrand must be identically zero. Also one may verify that any virtual velocity may be obtained by some variation of curves, so that indeed extremals satisfy d'Alembert and hence are trajectories of the system. □

For a functional given as above in 'integral form': $A(\gamma) = \int L(\gamma, \dot{\gamma}) \; dt$, we call the function $L$ the *Lagrangian* for the variational problem. For mechanical systems, it is often written with the force function $U = -V$ in place of the potential $V$. The value of the functional on a curve, $\int_\gamma L(\gamma, \dot{\gamma}) \; dt$, is called the *action* of the curve. The variational principle is often called the *principal of least action*, however one should be careful with this name in that extremals are not necessarily minimizers (this requires further verification), although any minimizer is an extremal.

Like d'Alembert's principle, the variational principle is convenient for writing down equations of motion in general coordinate systems. For this we have:

**Euler-Lagrange equations:** Let the configurations of a given system be (locally) parametrized by the variables $x = (x_1, ..., x_k) \in \mathbb{R}^k \to f(x) = q \in \Sigma$. Then $\dot{q} = d_x f(\dot{x})$ and upon expressing the Lagrangian in the new variable and their velocities, $L(x, \dot{x})$, the equations of motion in these coordinates have the form:

$$\frac{d}{dt} \left( \partial_{\dot{x}_j} L \right) = \partial_{x_j} L.$$

*proof:* The trajectories are still extremals of $x \mapsto \int L(x, \dot{x}) \; dt$. Hence, for a trajectory $x(t)$, with variation $x_\varepsilon$ and $\delta x = \frac{d}{d\varepsilon}|_{\varepsilon=0} x_\varepsilon$, we have from integration by parts:

$$0 = \int_{t_0}^{t_1} \partial_{\dot{x}} L \cdot \dot{\delta x} + \partial_x L \cdot \delta x \; dt = \int_{t_0}^{t_1} \left( \partial_x L - \frac{d}{dt} \partial_{\dot{x}} L \right) \cdot \delta x \; dt$$

for *any* vector field $\delta x$ along the trajectory $x$ which vanishes at its endpoints. Using a bump function, one may show that the integrand must be zero, ie the announced equations. □

Some further useful properties of variational problems of our type are:

**Energy conservation:** Consider a variational problem $x \mapsto \int_{t_o}^{t_1} L(x, \dot{x}) \; dt$ over curves $x : [t_o, t_1] \to \mathbb{R}^k$ with fixed endpoints. Then the *energy*:

$$E = \partial_{\dot{x}} L \cdot \dot{x} - L$$

is constant over extremals.

*proof:* The extremals satisfy the Euler-Lagrange equations: $\frac{d}{dt} \partial_{\dot{x}} L = \partial_x L$. We compute:
$\dot{E} = \partial_x L \cdot \dot{x} + \partial_{\dot{x}} L \cdot \ddot{x} - \partial_x L \cdot \dot{x} - \partial_{\dot{x}} L \cdot \ddot{x} = 0.$ □

We also have:

**Lagrange multipliers v1:** Let $\Gamma = \{\gamma : [t_o, t_1] \to \mathbb{R}^k\}$ consist of smooth curves with fixed endpoints and $\Gamma_o \subset \Gamma$ a 'subclass' of curves defined by a condition of the form $B = cst.$ for $B : \Gamma \to \mathbb{R}$. Consider a functional $A : \Gamma \to \mathbb{R}$.
If $\gamma_* \in \Gamma_o$ is an extremal of $A + \lambda B : \Gamma \to \mathbb{R}$ for some $\lambda \in \mathbb{R}$ then $\gamma_* \in \Gamma_o$ is an extremal of $A|_{\Gamma_o} : \Gamma_o \to \mathbb{R}$.

**Lagrange multipliers v2:** Let $\Gamma = \{\gamma : [t_o, t_1] \to \mathbb{R}^k\}$ consist of smooth curves with fixed endpoints and $\Gamma_o \subset \Gamma$ defined by a condition of the form $b(\gamma(t), \dot\gamma(t), t) = cst.$ where $b : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$. Consider a functional $A : \Gamma \to \mathbb{R}$.
If $\gamma_* \in \Gamma_o$ is an extremal of $\Gamma \ni \gamma \mapsto A(\gamma) + \int_{t_o}^{t_1} \lambda(t) b(\gamma, \dot\gamma, t)\ dt$ for some $\lambda : \mathbb{R} \to \mathbb{R}$ then $\gamma_* \in \Gamma_o$ is an extremal of $A|_{\Gamma_o} : \Gamma_o \to \mathbb{R}$.

Examples:

- Extremals of the length functional: $c(t) = (x(t), y(t)) \mapsto \int_0^1 \sqrt{\dot x^2 + \dot y^2}\ dt = \int_0^1 |\dot c|\ dt$ over fixed endpoint curves satisfy the Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\dot x}{\sqrt{\dot x^2 + \dot y^2}}\right) = 0 = \frac{d}{dt}\left(\frac{\dot y}{\sqrt{\dot x^2 + \dot y^2}}\right) \Rightarrow cst. = \frac{\dot y}{\dot x} = \frac{dy}{dx} \Rightarrow y = ax + b$, so the extremals are lines.

  Observe that one may also consider extremals of the 'energy functional', $c(t) \mapsto \int_0^1 \frac{1}{2}|\dot c|^2\ dt$, for which the Euler-Lagrange equations have the form: $\frac{d}{dt}\dot x = 0 = \frac{d}{dt}\dot y \Rightarrow \ddot x = \ddot y = 0$ and the extremals are again lines.

  To prove (as we would hope!) that the lines are not merely extremals but in fact minimizers, one may apply the following somewhat general method: let $q_o$ be a fixed point and suppose that for each $q$ one has determined a 'candidate minimizer', $\gamma_q : q_o \to q$. Define the function $S(q) := A(\gamma_q)$. Then if it holds that $S(q_o) = 0$ and $\Phi(q, \dot q) = d_q S(\dot q) \le L(q, \dot q)$ then $\gamma_q$ are all in fact minimizers. To prove this, one computes that for any curve $\gamma : q_o \to q$ one has $A(\gamma) = \int_\gamma L(\gamma, \dot\gamma)\ dt \ge \int_\gamma \Phi(\gamma, \dot\gamma)\ dt = S(q) - S(q_o) = A(\gamma_q)$.

  Now, for lines, we take $q_o$ for the origin and $\gamma_q : q_o \to q$ as $\gamma_q(t) = tq$ so that $S(q) = |q|$ and $d_q S(\dot q) = \frac{q \cdot \dot q}{|q|} = \Phi(q, \dot q)$. Now, by Cauchy-Schwarz, $\Phi(q, \dot q) \le L(q, \dot q) = |\dot q|$ and so indeed, lines minimize the length functional.

- The *brachistochrone* curve: let $(0,0), (x_o, y_o)$ be two points in the plane (with $y_o < 0$). For planar curves connecting these two points assign the time it takes a 'bead' to fall along the curve from point to point (subject to constant vertical acceleration $(0, -g)$). Then $\frac{|v|^2}{2} + gy = cst.$ by energy conservation, so that for the bead falling along the curve, we have: $|v|^2 = -2gy$ (recall $y_o < 0$). When the curve is given by a graph, $(x, y(x))$, we have then:

  $$y(x) \mapsto \int_0^\ell \frac{ds}{|v|} = \int_0^{x_o} \sqrt{\frac{1 + (y')^2}{-2gy}}\ dx$$

  is our variational problem. One could at this point write out the Euler-Lagrange equations (a 2nd-order ode for $y(x)$), however it is simpler to use the energy integral (a 1st-order ode for $y(x)$, depending on a constant parameter). Namely, we have:

  $$cst. = y' \partial_{y'} L - L$$

  over extremals, which may be re-arranged as:

  $$(y')^2 = -\frac{k^2 + y}{y}, \quad k = cst.$$

This may be integrated explicitely, through the following substitutions: $\sqrt{\frac{-y}{k^2+y}}dy = -dx \overset{y=-k^2Y}{\Rightarrow} \sqrt{\frac{Y}{1-Y}}dY = \frac{dx}{k^2} \overset{Y=\sin^2\frac{\theta}{2}}{\Rightarrow} \frac{k^2}{2}(1-\cos\theta)\ d\theta = dx \Rightarrow x = \frac{k^2}{2}(\theta - \sin\theta), y = -\frac{k^2}{2}(1-\cos\theta)$. So the extremals are cycloids!

- The *catenary* curve: let $(x_o, y_0), (x_1, y_1)$ two points in the plane (with $x_0 < x_1$ and $y_j > 0$) and consider the form of a hanging chain between these two points. For chains given by graphs $(x, y(x))$, the length of the chain is fixed, $\ell = \int_{x_o}^{x_1} \sqrt{1+(y')^2}dx$ and the potential energy of the chain is given (upto constant multiples) by $\int_{x_o}^{x_1} y\sqrt{1+(y')^2}dx$. By Lagrange multipliers, we seek extremals of:

$$y(x) \mapsto \int_{x_o}^{x_1} (y-\lambda)\sqrt{1+(y')^2}\ dx$$

over $y : [x_o, x_1] \to \mathbb{R}$ with $y(x_j) = y_j$ fixed, and $\lambda \in \mathbb{R}$ some constant. By energy conservation, such extremals satisfy $cst. = y'\partial_{y'}L - L$ which may be rewritten as:

$$\left(\frac{y-\lambda}{k}\right)^2 = 1 + (y')^2$$

for some constant $k \in \mathbb{R}$. The substitution $y - \lambda = k\cosh u$ leads to the equation: $y = k\cosh\frac{x+a}{k} + \lambda$ for the extremals ($a, k, \lambda$ constants).

Observe that another method to arrive at the catenary curves is to use the 2nd version of Lagrange multipliers above, applied to parametrized curves $(x(s), y(s)) : [0, \ell] \to \mathbb{R}^2$ with fixed endpoint conditions and the constraint that the parametrizations are by arc-length: $(x')^2 + (y')^2 = 1$.

- A free particle on a surface (a geodesic) may thus as well be characterized as an extremal of the kinetic energy functional: $q \mapsto \int \frac{1}{2}|\dot{q}|^2\ dt$, or equivalently extremals of the length functional: $q \mapsto \int |\dot{q}|\ dt$.

Exercises:

1. Give an example of a variational problem for which:

   (a) there are two points connected by multiple extremals,

   (b) there are two points connected by no extremals.

2. Consider a variational problem in Lagrangian form: $\gamma \mapsto \int L(\gamma, \dot{\gamma})\ dt$. Determine the conditions satisfied by the extremals when the class of curves, $\Gamma$, is:

   (a) $\Gamma = \{\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma$ is smooth and $\gamma(0) = \gamma(1)\}$

   (b) $\Gamma = \{\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma$ is smooth and $\gamma(0) \in \ell_0, \quad \gamma(1) \in \ell_1\}$ where $\ell_j$ are two fixed lines in the plane.

3. (a) For $f, g : [0,1] \to \mathbb{R}$ continuous, let $\langle f, g\rangle := \int_0^1 f(x)g(x)\ dx$. Show that this defines an inner product on the space of such functions, and, in particular, satisfies the Cauchy-Schwarz inequality.

   (b) Consider the two functionals $\ell(c) = \int_0^1 |\dot{c}|\ dt,\ E(c) = \int_0^1 |\dot{c}|^2\ dt$ defined over smooth plane curves $c : [0,1] \to \mathbb{R}^2$ with fixed endpoints. Show that: $c_*$ minimizes $E \iff c_*$ minimizes $\ell$ and $|\dot{c}_*| = cst$.

4. Prove the two versions of Lagrange multipliers for variational problems stated above (pg. 26).

5. Consider a variational problem defined through a time dependent Lagrangian: $\gamma \mapsto \int L(\gamma, \dot{\gamma}, t)\ dt$, over curves $\gamma : [0,1] \to \mathbb{R}^k$ with fixed endpoints. Show that along extremals, we have:

$$\frac{d}{dt}E = -\partial_t L$$

   where $E := \partial_{\dot{\gamma}} L \cdot \dot{\gamma} - L$ is the energy.

6. Consider surfaces of revolution generated by revolving planar curves of fixed length connecting two given points in the $xz$-plane about the $z$-axis. Describe the extremals of the functional sending such a curve to the area of the corresponding surface of revolution (called minimal surfaces of revolution).

7. Determine the form of a 'loaded chain', that is a hanging chain between two points $(x_o, y_o), (x_1, y_1)$ for which the mass from $(x_o, y_o)$ to $(x(s), y(s))$ is proportional to $x(s) - x_o$.

## §7 Noether's theorem

In the Lagrangian formalism, certain first integrals correspond to symmetries of the system. Essentially it amounts to the observation in coordinates that if $L(x, \dot{x})$ does not depend on some $x_k$ (a so-called *cyclic coordinate*), then, by the E-L equations, $\partial_{\dot{x}_k} L$ is a conserved quantity.

A *symmetry* of a Lagrangian system $L(q, \dot{q})$, $q \in Q$, is a transformation $\varphi : Q \to Q$ preserving the Lagrangian: $L(\varphi(q), d_q\varphi(\dot{q})) = L(q, \dot{q})$. Observe that a symmetry sends solutions to solutions (since it preserves action and so extremals).[1] Now, a *1-parameter family* of symmetries consists of a symmetry $\varphi_s : Q \to Q$ for each $s \in \mathbb{R}$ with $\varphi_{s_1+s_2} = \varphi_{s_1} \circ \varphi_{s_2}$ (in particular $\varphi_o = id$). Then:

**Noether's theorem v1:** Suppose the Lagrangian system, $L(q, \dot{q})$, admits a 1-parameter family, $\varphi_s$, of symmetries. Then it admits the first integral:

$$\partial_{\dot{q}} L(q, \dot{q}) \cdot X(q) = \partial_{\dot{q}} L \cdot X$$

where $X(q) = \frac{d}{ds}|_{s=0}\varphi_s(q)$.

*proof:* Let $\hat{\varphi}_s(q, v) = (q_s(q, v), v_s(q, v))$ be a transformation of $TQ$ preserving action of trajectories. For $(q(t), \dot{q}(t))$ a solution of the system, we have $v_s(q(t), \dot{q}(t)) = \dot{q}_s(q(t), \dot{q}(t))$ since $\hat{\varphi}$ takes solutions to solutions. Now:

$$0 = \frac{d}{ds}|_{s=0} \int_{t_0}^{t_1} L(\hat{\varphi}_s(q(t), \dot{q}(t)))\ dt = \int_{t_o}^{t_1} \partial_q L \cdot X + \partial_{\dot{q}} L \cdot \dot{X}\ dt$$

where $X = \frac{d}{ds}|_{s=0} q_s$. Integration by parts and the Euler-Lagrange equations gives:

$$0 = (\partial_{\dot{q}} L \cdot X)\,|_{t_o}^{t_1}.$$

$\square$

From the proof, one sees that in fact one obtains conserved quantities for slightly more general symmetries than those we have defined here, namely for action preserving transformations of the tangent bundle $TQ$. In fact 'Noether's theorem' consists of various generalizations of what we have presented here (as well to field theories –pde's– defined by Lagrangian densities). We will finish here with a generalization to include possible time dependence.

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ define a Lagrangian system which may depend on time. A symmetry of such a system is a transformation $\varphi : Q \times \mathbb{R} \to Q \times \mathbb{R}$ such that $\hat{L} \circ \hat{\varphi} = \hat{L}$, where $\hat{L} : T(Q \times \mathbb{R}) \to \mathbb{R}, (q, t, q', t') \mapsto L(q, q'/t', t)t'$ and $\hat{\varphi} : T(Q \times \mathbb{R}) \to T(Q \times \mathbb{R}), (q, t, q', t') \mapsto (\varphi(q, t), d_{(q,t)}\varphi(q', t'))$.

As before, we may consider 1-parameter families by such symmetries, and have:

**Noether's theorem v2:** Suppose the Lagrangian system, $L(q, \dot{q}, t)$ admits a 1-parameter family, $\varphi_s$, of symmetries. Then it admits the first integral:

$$\partial_{\dot{q}} L \cdot X + T(L - \dot{q} \cdot \partial_{\dot{q}} L)$$

where $X = \frac{d}{ds}|_{s=0}\pi_1 \circ \varphi_s(q, t)$ and $T = \frac{d}{ds}|_{s=0}\pi_2 \circ \varphi_s(q, t)$ for $\pi_1 : Q \times \mathbb{R} \to Q, \pi_2 : Q \times \mathbb{R} \to \mathbb{R}$ the standard projections.

*proof:* The 1-parameter family, $\varphi_s$, are symmetries of the Lagrangian system $\hat{L}$ in the sense of Noether's theorem v1 we proved above. Hence:

$$\partial_{(q',t')} \hat{L} \cdot \hat{X} = \partial_{\dot{q}} L \cdot X + T(L - \partial_{\dot{q}} L \cdot \dot{q})$$

[1] However the converse is not necessarily true, namely there may be more general transformations sending solutions to solutions which fail to preserve $L$. For example, free particles: $L = \frac{|\dot{q}|^2}{2}$, $q \in \mathbb{R}^k$, are uniform straight line motions, $\ddot{q} = 0 \Rightarrow q(t) = q_o + v_o t$. Any invertible linear transformation $A \in \mathrm{GL}(\mathbb{R}^k)$ sends solutions to solutions, but unless $A \in \mathrm{O}_k$ (preserving Euclidean norm), we do not have that $(q, \dot{q}) \mapsto (Aq, A\dot{q})$ preserves $L$ (eg scaling $(q, \dot{q}) \mapsto \lambda(q, \dot{q})$, $\lambda \in \mathbb{R}$ has $L \mapsto \lambda^2 L$).

is a conserved quantity over extremals of $\hat{L}$ in $T(Q\times\mathbb{R})$. Now, observe that for a curve $(q(\tau),t(\tau))\in Q\times\mathbb{R}$ of $\hat{L}$, (with $t'=\frac{dt}{d\tau}\neq 0$) we have

$$\int_{\tau_o}^{\tau_1}\hat{L}(q,t,q',t')\;d\tau=\int_{t_o}^{t_1}L(q,\dot{q},t)\;dt$$

so that extremals of $\hat{L}$ in $T(Q\times\mathbb{R})$ are given by lifts $(q,t,\dot{q},1)$ of extremals of $L$, and so as well we have a first integral over the extremals of $L$. □

In particular this last variant of Noether's theorem may be applied to a time independent Lagrangian system to 'explain' the energy integral: it corresponds to the time-translation symmetry.

Often, symmetries of a system come not just in 1-parameter families but from differentiable group actions (Lie group actions). That is, one has an action $q\mapsto g\cdot q$ of a Lie group $G$ on $Q$ by symmetries of the system. In this case, one has for each $\xi\in\mathfrak{g}$,[1] the first integral

$$\partial_{\dot{q}}L\cdot X_\xi$$

with $X_\xi=\frac{d}{ds}|_{s=0}g(s)\cdot q$, where $\xi=\frac{d}{ds}|_{s=0}g(s)$.

Observe that Noether's theorem gives conserved quantities (connected to symmetries) more power, and can be used to 'explain' how certain systems are 'solvable by integrals'. Let us briefly revisit central force problems: $\ddot{q}=-\frac{f(r)}{r}q$, with $q\in\mathbb{R}^2$. We seek integral curves of a vector field on the 4-dimensional position-velocity space or *state space*: $T\mathbb{R}^2=\mathbb{R}^2\times\mathbb{R}^2$. The energy and angular momentum (rotational symmetry) are first integrals, so we may fix their values and seek integral curves on a 2-dimensional invariant set, $\Sigma_{e,c}$ (which we know to typically be a torus). The symmetry by rotations tells us it suffices to find *one* such solution curve, since solutions are sent to solutions by the symmetry. In other words, if the symmetry preserves the level sets of the corresponding first integrals, then one may pass to a quotient by the symmetry action, $\bar{\Sigma}_{e,c}=\Sigma/\{x\sim\varphi_s(x)\}$ in this case a vector field on a one-dimensional space (solutions of which are always expressed by an integral).

So the connection between symmetries and integrals is a powerful observation in understanding how one may expect to reduce the dimensions of a problem under consideration. However, it does not always associate *all* integrals to symmetries, for example the additional integrals in the Hooke central force problem are not all associated to Noether symmetries. A generalization of Noether's theorem to associate all first integrals to symmetries is best seen and stated in the structure of Hamiltonian mechanics (Poisson brackets).

Examples:

- Suppose $L(q,\dot{q})$, $q\in\mathbb{R}^{3N}$ admits *translational symmetries*: $\varphi(q)=q+(a,...,a)$ for $a\in\mathbb{R}^3$. When it admits the 1-parameter family, $\varphi_s(q)=q+s(a,...,a)$, we have the conserved quantity: $\sum\partial_{\dot{q}_j}L\cdot a$.

  When the system is *natural*: $L(q,\dot{q})=\sum\frac{m_j|\dot{q}_j|^2}{2}+U(q)$, then $(\sum m_j\dot{q}_j)\cdot a$ is conserved. If the system admits translational symmetries for any $a\in\mathbb{R}^3$ then linear momentum, $\sum m_j\dot{q}_j$, is conserved.

- Suppose $L(q,\dot{q}), q\in\mathbb{R}^{3N}$ admits *rotational symmetries*: $\varphi(q)=(Aq_1,...,Aq_N)$ for $A\in\mathrm{SO}_3$. When it admits the 1-parameter family, $\varphi_s(q)=(R_sq_1,...,R_sq_N)$ for $R_s$ rotation by angle $s$ about the axis $\vec{\omega}$, we have the conserved quantity: $\vec{\omega}\cdot\left(\sum q_j\times\partial_{\dot{q}_j}L\right)$.

  When the system is natural, then $\vec{\omega}\cdot(\sum q_j\times m_j\dot{q}_j)$ is conserved. If the system admits rotational symmetries for any $A\in\mathrm{SO}_3$ then angular momentum, $\sum q_j\times m_j\dot{q}_j$, is conserved.

- The free rigid-body admits translational and rotational symmetry, hence by Noether's theorem the linear and angular momentum are conserved (as we derived explicitely above).

  In more detail, by fixing a 'reference configuration' $B$, the configurations of the rigid body are identified with $(g,q_o)\in\mathrm{SO}_3\times\mathbb{R}^3\cong Q\subset\mathbb{R}^{3N}$, $(g,q_o)\leftrightarrow b=\{q=gQ+q_o:Q\in B\}$, and $\dot{q}=\dot{q}_o+\omega(q-q_o)$ for $\omega=\dot{g}g^{-1}$.

[1] The Lie algebra, $\mathfrak{g}$, of $G$, may be realized as the tangent space to $G$ at the identity: $\xi=\frac{d}{ds}|_{s=0}g(s)$ for some smooth curve $g(s)\in G$ with $g(0)=e$.

Since the body is free, it is represented by the Lagrangian system: $L(g, q_o, \dot{g}, \dot{q}_o) = \int_b |\dot{q}|^2 \, d\mu$.

The action by translations is by $q \mapsto q + a$, $a \in \mathbb{R}^3$, with lift $(q, \dot{q}) \mapsto (q + a, \dot{q})$, which preserves $L$ since: $\int_{b+a} |\dot{q}|^2 \, d\mu = \int_b |\dot{q}|^2 \, d\mu$.

The action by rotations is by $q \mapsto hq$, $h \in \mathrm{SO}_3$, with lift $(q, \dot{q}) \mapsto (hq, h\dot{q})$, which preserves $L$ since: $\int_{h(b)} |h\dot{q}|^2 \, d\mu = \int_{h(b)} |\dot{q}|^2 \, d\mu = \int_b |\dot{q}|^2 \, d\mu$.

Observe that upon fixing the center of mass as the origin, the rotational motion of the rigid body is described as extremals of the Lagrangian system $L(g, \dot{g}) = \int_{q \in b} |\dot{g}g^{-1}q|^2 \, d\mu = \int_{Q \in B} |\dot{g}Q|^2 \, d\mu$ on $T\mathrm{SO}_3$. In other words, as geodesics of a *left-invariant metric* on $\mathrm{SO}_3$: $L(hg, h\dot{g}) = L(g, \dot{g})$.

- Now we will consider a new example, the *Lagrange top*: a rigid body symmetric about an axis is subject to uniform gravity and we study its motion when one point on its symmetry axis is fixed (the rigid body is free to pivot about this point). In practice one can imagine that the body is a 'spinning top' and the fixed point is achieved by placing the contact point in small dent in the table.

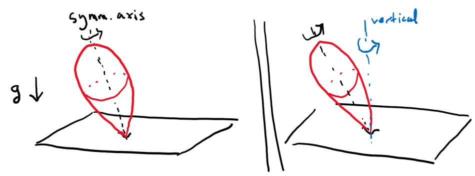


Figure 21. A Lagrange top. The system admits two 1-parameter symmetries: rotations about the symmetry axis of the top, and rotations about the vertical direction.

Taking the fixed point along the symmetry axis as origin, and a reference configuration, all configurations of the top are given by elements of $\mathrm{SO}_3$. Before deriving any equations, we know that the forces are conservative so energy is conserved. As well we have two symmetry actions: rotation about the current position of the symmetry axis of the body, and another by rotations about the vertical axis. By Noether's theorem, there are two additional first integrals. Hence the problem will admit three first integrals of motion.

Fixing the values of these three first integrals we expect in general to allow us to restrict our attention to a vector field on some 3-dimensional space inside of the 6-dimensional state space: $T\mathrm{SO}_3$. However the presence of a 2-parameter family of symmetries (taking solutions to solutions) we expect to pass us to a 'quotient' of this 3-dimensional space by this 2-dimensional action, finally arriving at a vector field along a 1-dimensional object, solvable upto integrals!

Hence without any computation, we have good reason to expect that the Lagrange top will be a 'solvable' system. Now, let us get down to business and derive some equations. We will use coordinates on $\mathrm{SO}_3$ called *Euler angles*: $(\psi, \theta, \varphi) \mapsto R_3(\varphi)R_1(\theta)R_3(\psi) \in \mathrm{SO}_3$ (see figure).

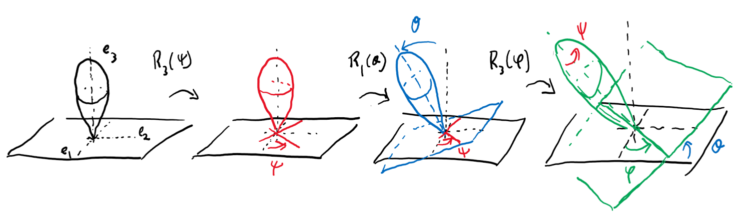


Figure 22. We take a vertical reference configuration: with the symmetry axis of the top vertical. Let $e_1, e_2, e_3$ be principal axes of the top for this reference configuration (with moments of inertia $I = I_1 = I_2$ and $I_3$ resp.). Then the Euler angles parametrize rotations as a certain composition of rotations about these fixed principal axes.

The potential energy is: $V = \int gz \; d\mu$ which is the $z$-component of the center of mass multiplied by the total mass, $M$, of the rigid body. Let $\ell$ by the distance from the center of mass of the rigid body (along the symmetry axis) to the fixed point, we have:

$$V = gM\ell \cos\theta.$$

As for the kinetic energy, for $\mathbb{I}_o$, the inertia tensor of the reference configuration, we have $K = \frac{1}{2}\mathbb{I}_o\vec{\Omega}\cdot\vec{\Omega}$, where $\vec{\Omega}$ corresponds to $g^{-1}\dot{g} = \Omega \in \mathfrak{so}_3$. Now, by our symmetry observation, $K$ does not depend on $\psi, \varphi$, and indeed we compute:

$$R_3(\psi)\vec{\Omega} = \dot{\varphi}R_1(-\theta)e_3 + \dot{\theta}e_1 + \dot{\psi}e_3$$

$$= \dot{\theta}e_1 + \dot{\varphi}\sin\theta e_2 + (\dot{\psi} + \dot{\varphi}\cos\theta)e_3$$

for $e_1, e_2, e_3$ the principal axis of the rigid body (with moments of inertia $I_1 = I_2 = I, I_3$), so:

$$K = \frac{I}{2}\left(\dot{\theta}^2 + \sin^2\theta \; \dot{\varphi}^2\right) + \frac{I_3}{2}\left(\dot{\psi} + \cos\theta \; \dot{\varphi}\right)^2.$$

So we have a Lagrangian, $L = K - V$, and may determine the equations of motion (E-L eqs.), as well as the first integrals in these coordinates:

$$C_{ax} = \partial_{\dot{\psi}}L = \mathbb{I}_o\vec{\Omega}\cdot e_3 = I_3\Omega_3,$$

$$C_{ver} = \partial_{\dot{\varphi}}L = \mathbb{I}_o\vec{\Omega}\cdot(\sin\theta e_2 + \cos\theta e_3) = I\Omega_2 \; \sin\theta + C_{ax} \; \cos\theta.$$

By fixing these integrals, we find the energy integral expressed as:

$$E = \frac{I}{2}\dot{\theta}^2 + \frac{1}{2I\sin^2\theta}\left(C_{ver} - C_{ax} \; \cos\theta\right)^2 + \frac{C_{ax}^2}{2I_3} + gM\ell\cos\theta,$$

which, while complicated, is of the form $E = \frac{I}{2}\dot{\theta}^2 + V_{eff}(\theta)$, so may be analyzed and 'solved' by the same method of effective potentials we applied to central force problems! This –at least in principle determines– $\theta(t)$ explicitely. A scheme for the full solution would be then to find $\varphi(t)$ from the 1st order ode $C_{ver}(\theta(t), \dot{\varphi}) = cst.$ and then finally $\psi(t)$ from the ode $C_{ax}(\dot{\psi}, \dot{\varphi}(t), \theta(t)) = cst.$. One is typically content to describe the motion of the symmetry axis $(\theta(t), \varphi(t))$. In practice, one describes the motion by succesive methods of effective potentials for each first order system rather than solving them explicitely. Also, one may describe motions near equilibrium points by the method of small oscillations (next section).

**§8 small oscillations**

The Lagrangian formalism is useful for performing computations to 'linearize' systems around certain orbits. Such linearizations are simpler systems of equations approximating the local behaviour of the system.

First we will consider the local behaviour of Lagrangian systems around equilibrium points. An *equilibrium point* of a Lagrangian system $L : TQ \to \mathbb{R}$ is a constant solution of the E-L equations: $q(t) \equiv q_o, \dot{q}(t) \equiv 0$ for all time.

More generally, a vector field $v(x)$ on $\mathbb{R}^k$ with corresponding ode $\dot{x} = v(x)$, has equilibrium points at zeroes of $v$: $v(x_o) = 0$, so that $x(t) \equiv x_o$ is an equilibrium solution. The *linearization* of the system $\dot{x} = v(x)$ around the equilibrium $x_o$ is then defined as the system of ode's:

$$\dot{X} = (d_{x_o} v)\, X, \quad X \in \mathbb{R}^k$$

where $d_{x_o} v$ is the Jacobian matrix of $\mathbb{R}^k \ni x \mapsto v(x) \in \mathbb{R}^k$ at $x_o$.

The linearization of a system approximates the true motions of the original system near $x_o$ in the following sense: let $x(t) = x_o + \varepsilon X(t)$ be a solution of the original system. Then:

$$\varepsilon \dot{X} = \dot{x} = v(x) = v(x_o + \varepsilon X) = \varepsilon\,(d_{x_o} v)\, X + O(\varepsilon^2)$$

$$\Rightarrow \dot{X} = (d_{x_o} v)\, X + O(\varepsilon).$$

Hence, as $\varepsilon \to 0$, the vector $X(t)$ tends to a solution of the linearized system. More precisely:

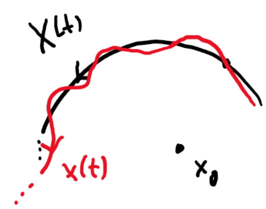


Figure 23. Over *finite* time scales, solutions to the linearized system $X(t)$ stay close to true solutions $x(t)$ provided they start with initial conditions sufficiently close to the equilibrium $x_o$.

**Linear approximation:** Let $x_o$ be an equilibrium point of the vector field $v(x)$. Then for any $\varepsilon, T > 0$, there exists a $\delta > 0$ such that for solutions $x(t), X(t)$ of $\dot{x} = v(x), \dot{X} = (d_{x_o} v)\, X$ with $x(0) = X(0)$ and $|x(0) - x_o| < \delta$ it holds that:

$$|x(t) - X(t)| < \varepsilon, \quad t \in [0, T].$$

*proof:* Wlog, suppose $x_o = 0$ and let $\varepsilon, T > 0$ be given. By continuous dependence on initial conditions, for each $\delta_* > 0$ we have $M(\delta_*) := \sup_{|x(0)| \le \delta_*} \{|x(t)|^2, t \in [0,T]\} \in (0, \infty)$ such that $|x(t)|^2 \le M(\delta_*)$ for $t \in [0, T]$ and any solution $x(t)$ with $|x(0)| < \delta_*$. Note that $M(\delta_*)$ is continuous in $\delta_*$ with $M(0) = 0$. Set $z(t) = x(t) - X(t) = \int_0^t \dot{x}(s) - \dot{X}(s)\, ds = \int_0^t Lz(s) + O(x(s)^2)\, ds$, where $L := d_{x_o} v$ and $x(t), X(t)$ are as in the theorem with $|x(0)| < \delta_* \le 1$. Then:

$$|z(t)| \le kM(\delta_*)t + \int_0^t |L||z(s)|\, ds, \quad t \in [0, T]$$

where $|L| = \sup_{x \neq 0} \frac{|Lx|}{|x|} \in (0, \infty)$ is the matrix norm of $L$ and $k = \sup_{|x| < M(1)} |d_x^2 v|$ is some constant. Now, by Gronwall's lemma[1], we obtain:

$$|z(t)| \le \frac{kM(\delta_*)}{|L|} \left( e^{|L|T} - 1 \right) = CM(\delta_*), \quad t \in [0, T]$$

[1] For $c, u, v$ positive (differentiable) functions on $[0, T]$ with $c(0) = 0$ and satisfying $v(t) \le c(t) + \int_0^t u(s)v(s)\, ds$ one has $v(t) \le \int_0^t c'(s) e^{\int_s^t u(\tau)\, d\tau} ds$. Here we apply the lemma with $v(t) = |z(t)|, u(t) = |L|, c(t) = kM(\delta_*)t$.

for $C > 0$ some constant. Now take $\delta > 0$ s.t. $M(\delta) < \varepsilon/C$ (continuity of $M(\delta_*)$ and $M(0) = 0$). $\square$

For Lagrangian systems of the form $L(q, \dot{q}) = \frac{(A(q)\dot{q})\cdot\dot{q}}{2} + U(q)$ with $A(q)$ a (possibly $q$-dependent) symmetric positive definite matrix, the E-L equations read:

$$\frac{d}{dt}A(q)\dot{q} = \partial_q U + \partial_q A\dot{q}\cdot\dot{q}$$

and the equilibrium points, $q_o$, are exactly the critical points of $U$: $\partial_q U(q_o) = 0$ (with zero velocities). The linearization of such systems may be found as follows. Let $q = q_o + \varepsilon Q, \dot{q} = \varepsilon\dot{Q}$. Set $A := A(q_o)$ and $B := \partial_q^2 U(q_o)$ the Hessian matrix of $U$ at $q_o$. Then the E-L equations may be written:

$$\frac{d}{dt}\left(\varepsilon A\dot{Q} + O(\varepsilon^2)\right) = \partial_q U(q_o + \varepsilon Q) + O(\varepsilon^2) = \varepsilon BQ + O(\varepsilon^2) \Rightarrow A\ddot{Q} = BQ + O(\varepsilon),$$

and the linearized system is $\ddot{Q} = \left(A^{-1}B\right)Q$.

Alternately, this linearized system may be derived somewhat more simply by working directly with the Lagrangian. We have:

$$L = \varepsilon^2\frac{A(q)\dot{Q}\cdot\dot{Q}}{2} + U(q_o + \varepsilon Q) = \frac{\varepsilon^2}{2}\left(A\dot{Q}\cdot\dot{Q} + BQ\cdot Q\right) + U(q_o) + O(\varepsilon^3)$$

so that the E-L equations are $\varepsilon^2 A\ddot{Q} = \varepsilon^2 BQ + O(\varepsilon^3)$, or $A\ddot{Q} = BQ + O(\varepsilon)$ as before.

The linearized system is thus also a Lagrangian system with Lagrangian: $L_o := \frac{1}{2}\left(A\dot{Q}\cdot\dot{Q} + BQ\cdot Q\right)$, where $A = A(q_o)$ is symmetric and positive definite and $B = \partial_q^2 U(q_o)$ is symmetric. We recall from linear algebra that $A$ and $B$ may be simlutaniously diagonalized: $\exists P$ s.t. $P^T AP = id, P^T BP = \Lambda$ with $\Lambda = diag(\lambda_1, ..., \lambda_n)$, $\lambda_j \in \mathbb{R}$ a diagonal matrix. Let $Q = Px, \dot{Q} = P\dot{x}$. Then the linearized Lagrangian is:

$$L_o = \frac{1}{2}\left(\dot{x}\cdot\dot{x} + \Lambda x\cdot x\right)$$

with E-L equations:

$$\ddot{x}_j = \lambda_j x_j$$

for $\lambda_j \in \mathbb{R}$ –the *characteristic frequencies* of the system around $q_o$ – determined as the roots of:

$$0 = \det(B - \lambda A).$$

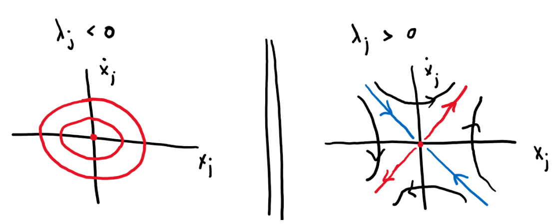


Figure 24. Qualitative properties of the linearized system are determined by the signs of the characteristic frequencies $\lambda_j$.

The stability type of an equilibrium point may in certain cases be determined from properties of the linearized system. An equilibrium point $q_o$ is called *stable* if for any neighborhood $U \ni q_o$ there exists a neighborhood $q_o \in V \subset U$ such that any trajectory $q(t)$ of the system with $q(0) \in V$ has $q(t) \in U$ for all time. An equilibrium is *unstable* when it is not stable. One can show: (1) if some $\lambda_j > 0$ in the linearized system (the linear system is unstable), then the true system is unstable. (2) If all $\lambda_j < 0$ then the equilibrium is stable (in the true system). When some $\lambda_j = 0$, it is necessary to take into account higher order terms to determine the behaviour of the true system.

Now, we turn our attention to linearized equations around trajectories (in particular periodic orbits). In general, let $x(t)$, be a solution to the ode $\dot{x} = v(x)$. We seek to describe behaviour of trajectories close to $x(t)$. Let $y(t) = x(t) + \varepsilon X(t)$ be a solution of the system, then:

$$\dot{y} = v(y) \Rightarrow \dot{X}(t) = \left(d_{x(t)}v\right) X(t) + O(\varepsilon).$$

The system $\dot{X} = A(t)X, \;\; X \in \mathbb{R}^k$ with $A(t) := d_{x(t)}v$ the Jacobians of $v$ along $x(t)$ are called the *variational equations* of the trajectory $x(t)$. As before, we have an analytic result that over finite time scales, solutions of the variational equations remain close to those of the true system, provided they have initial conditions sufficiently close to that of $x(t)$.

Note that direct study of the variational equations is much more difficult than that of linearized systems about equilibrium points: first of all one needs an explicit parametrization of the trajectory $x(t)$, and then one needs to solve a (linear) system with time dependent coefficients.

Let us proceed in general to see what information is contained in solutions to the variational equations. We will consider variational equations along a periodic orbit: $x(t+T) = x(t)$ for some $T > 0$. Then as well $A(t+T) = A(t)$. Consider a fundamental system, $X_1(t), ..., X_k(t)$, of solutions to the variational equations (so $X_j(0)$ is a basis of $\mathbb{R}^k$) with $X_k(0) = \dot{x}(0)$. Let $B = [X_1, ..., X_k]$ be the corresponding fundamental matrix (so the general solution to the variational equations is $X(t) = B(t)X(0)$).

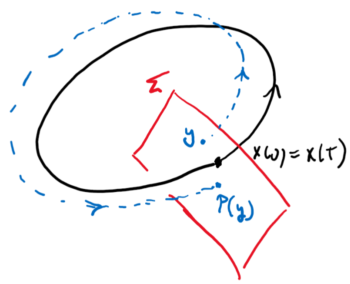


Figure 25. The Poincaré return map associated to a periodic orbit describes nearby trajectories.

A periodic orbit may be studied through its *Poincaré first return maps*: let $\Sigma$ be a hypersurface containing $x(0)$ and transverse to $\dot{x}(0)$. $P : \Sigma \to \Sigma$ is defined by sending $y \in \Sigma$ to the next intersection of the solution $y(t)$ to $\dot{y} = v(y)$ with initial condition $y(0) = y$ with $\Sigma$ ($P$ is always defined on some neigborhood of $x(0)$ in $\Sigma$). Note that $P(x(0)) = x(T) = x(0)$. Taking $\Sigma = \dot{x}(0)^{\perp}$, we have:

**Linearized Poincaré return map:** The linearization of the Poincaré return map at $x(0)$ is equal to the projection of $B(T)|_{\Sigma}$ to $\Sigma$, ie, $B(T) = \begin{pmatrix} d_{x(0)}P & 0 \\ * & 1 \end{pmatrix}$

*proof:* First note that $X_k(t) = \dot{x}(t)$, since $\dot{X}_k(t) = \frac{d}{ds}|_{s=0}\dot{x}(t+s) = \frac{d}{ds}|_{s=0}v(x(t+s)) = \left(d_{x(t)}v\right)\dot{x}(t) = \left(d_{x(t)}v\right)X_k(t)$. In particular, $B(T)X_k(0) = X_k(T) = X_k(0)$.
Now, let $\xi \in \dot{x}(0)^{\perp}$, and consider the trajectory with initial condition $x(0) + \varepsilon\xi$. Write this trajectory as $x(t) + \varepsilon\xi_{\varepsilon}(t)$ where $\xi_{\varepsilon}(t)$ satisfies $\dot{\xi}_{\varepsilon}(t) = A(t)\xi_{\varepsilon}(t) + O(\varepsilon)$. Then $P(x_o + \varepsilon\xi) = x(T_{\varepsilon}) + \varepsilon\xi_{\varepsilon}(T_{\varepsilon})$, where $T_{\varepsilon}$ is the time of return (note $T_o = T$ is the period of $x(t)$). Hence:

$$d_{x_o}P(\xi) = \frac{d}{d\varepsilon}|_{\varepsilon=0}P(x_o + \varepsilon\xi) = \dot{x}(T)T' + \xi_o(T) = T'X_k(0) + B(T)\xi$$

so that the matrix of $B(T)$ in the basis $X_j(0)$ (with $X_1(0), ..., X_{k-1}(0) \in \Sigma$ and $X_k(0) = \dot{x}(0)$) is as stated. □

As before, properties of the linearized Poincaré map may yield information about stability properties of the true system. A periodic orbit, $x(t)$, is called stable when for any neighborhood $U \supset \{x(t)\}$ of the orbit, there exists a neighborhood $V \subset U$ of the orbit with $y(t) \in U$ for all time whenever $y(0) \in V$. One may show that: if the linearized Poincaré map has an eigenvalue –called *Floquet multipliers*– with $|\lambda| \neq 1$ then the orbit is unstable (in the true system). Hence a necessary condition for stability is that all eigenvalues of the variational equations have norm 1 (called *linear stability* of the periodic orbit).

We will examine some particular properties of variational equations for orbits of Lagrangian systems when we consider the 'second variation' along trajectories in the following section.

EXAMPLES:

- Consider two *coupled pendulums*: the bobs interact with eachother (via a connecting spring):

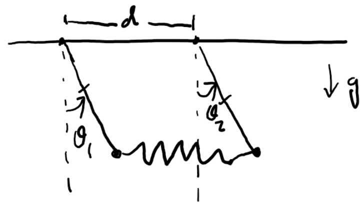


Figure 26. Coupled pendulums. The bobs interact via a connecting spring of rest length $d$.

For pendulums of equal length and masses (say 1), and the system in equilibrium with the bobs straight down take $\theta_1, \theta_2$ as the angles measured from the vertical. Then the kinetic energy is:

$$K = \frac{1}{2}\left(\dot\theta_1^2 + \dot\theta_2^2\right).$$

As for the potential energy, it is the sum of the gravitational (taking $g = 1$) potentials $1 - \cos\theta_j$ and a potential for the interaction force. The spring contributes the term $V_{int} = \frac{k}{2}(d_{12} - d)^2 = \frac{k}{2}(\theta_1 - \theta_2)^2 + O_3(\theta_j)$ to the potential, so

$$V = 1 - \cos\theta_1 + 1 - \cos\theta_2 + \frac{k}{2}(\theta_1 - \theta_2)^2 + O_3(\theta_j)$$

and we have $L = K - V$, with linearization around the bottom equilibrium point:

$$L_o = \frac{\dot\theta_1^2 + \dot\theta_2^2}{2} - \frac{(1+k)(\theta_1^2 + \theta_2^2) - 2k\theta_1\theta_2}{2}.$$

The linearized system is diagonalized by $Q_1 = \frac{\theta_1+\theta_2}{\sqrt{2}}, Q_2 = \frac{\theta_1-\theta_2}{\sqrt{2}}$, taking the form:

$$L_o = \frac{\dot Q_1^2 + \dot Q_2^2}{2} - \frac{Q_1^2 + \omega^2 Q_2^2}{2}$$

where $\omega = \sqrt{1+2k}$, so that $\ddot Q_1 = -Q_1, \ddot Q_2 = -\omega^2 Q_2$ and general solution $Q_1 = A_1\cos t + B_1 \sin t, Q_2 = A_2\cos\omega t + B_2\sin\omega t$ (see fig. 27).

When the strength of the coupling is very small, $0 < k << 1$, one may write the solution of the linearized system with initial conditions $Q_1(0) = Q_2(0) = 0, \dot Q_1(0) = \dot Q_2(0) = \frac{v}{\sqrt{2}}$ in the original variables as:

$$\theta_1 = v\sin\omega' t\cos\varepsilon t + O(k), \quad \theta_2 = v\cos\omega' t \sin\varepsilon t + O(k)$$

where $\omega' = \frac{\omega+1}{2} = 1 + O(k), \varepsilon = \frac{1-\omega}{2} = O(k)$ exhibiting 'exchange of energy' (see fig. 28).

- We consider a 'swing': the swing is modelled as a pendulum subject to external time dependent forces,

$$\ddot\theta = -\frac{g}{\ell}\sin\theta + f(\theta, t).$$

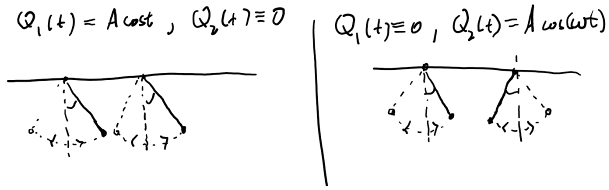


Figure 27. Eigenvalue solutions of the linearized equations or 'fundamental modes'. The pendulums oscillate perfectly in sync: $\theta_1(t) = \theta_2(t)$ or prefectly out of sync: $\theta_1(t) = -\theta_2(t)$.

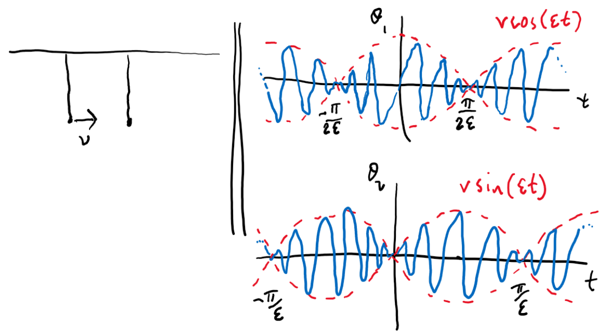


Figure 28. For small coupling strength, the linearized solutions exhibit exchange of energy over long time scales (see eg this video).

One imagines these external forces as being caused by our movements or 'pumping scheme' at the bottom of the swing. We assume that our motions are periodic, $f(\theta, t + T) = f(\theta, t)$ for some $T > 0$ and that when the swing is in its lowest point equilibrium, our motions have no effect, $f(0, t) \equiv 0$.

The system is represented by the first order ode's:

$$\dot{\theta} = v, \quad \dot{v} = -\frac{g}{\ell}\sin\theta + f(\theta, t), \quad \dot{t} = 1$$

on the space $(\theta, v, t) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{R}/T\mathbb{Z}$. There is an 'obvious' periodic orbit, $\theta(t) = v(t) = 0, t = t$ representing us pumping away with no effect exactly at the equilibrium position.

We apply linearization to study the stability behaviour of the swing near this periodic orbit. The variational equations[1] are:

$$\dot{X} = A(t)X, \quad X = \begin{pmatrix} \theta \\ \omega \end{pmatrix}, A(t) = \begin{pmatrix} 0 & 1 \\ -\omega(t)^2 & 0 \end{pmatrix}$$

where $\omega(t)^2 := \frac{g}{\ell} - \partial_\theta f(0, t)$. The general solution may be expressed as

$$X(t) = B(t)X_o, \quad B(t) = \begin{pmatrix} X_1(t) & X_2(t) \end{pmatrix}$$

where $X_j(t)$ are solutions with initial conditions $(1, 0), (0, 1)$. The linearized Poincaré return map ($t = 0$ to $t = T$) is thus given by

$$\begin{pmatrix} \theta \\ v \end{pmatrix} \overset{P}{\mapsto} B(T)\begin{pmatrix} \theta \\ v \end{pmatrix}.$$

Observe[2] that in this case $\det P = 1$. Hence the Floquet multipliers are the roots of:

$$0 = \lambda^2 - tr(P)\lambda + 1.$$

[1] Alternately, as a 2'nd order *Hill's equation*: $\ddot{\theta} = -\omega(t)^2\theta$.

[2] This follows by considering the *Wronskian*, $W(t) = \det B(t)$, which satisfies $\dot{W} = tr(A(t))W = 0$.

And the roots are real $\lambda_1 = \lambda_2^{-1} \in \mathbb{R}_*$, when $|tr(P)| > 2$ exhibiting instability, and complex $\lambda_1 = \bar{\lambda}_2^{-1} \in S^1$, when $|tr(P)| < 2$ giving linear stability.

Consider now when the linearized system has the form $\omega(t)^2 = \omega_o^2 + \varepsilon a(t)$, with $\varepsilon$ small and $a(t+T) = a(t)$ of period $T > 0$. Then for $\varepsilon = 0$, we have $P_0 = \begin{pmatrix} \cos\omega_o T & \frac{\sin\omega_o T}{\omega_o} \\ -\omega_o \sin\omega_o T & \cos\omega_o T \end{pmatrix}$, with $tr(P_0) = 2\cos\omega_o T$. Since the linearized Poincaré maps vary continuously in $\varepsilon$, we see that linear stability will continue to hold for small $\varepsilon$-values provided that $T \neq \frac{k\pi}{\omega_o}$ some $k \in \mathbb{N}$. These stable regions will in general bound an unstable region of $T$ and $\varepsilon$ values, which as $\varepsilon \to 0$ converge to $T = \frac{k\pi}{\omega_o}, k \in \mathbb{N}$.

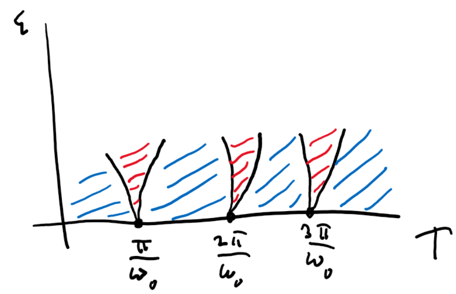


Figure 29. The oscillation period of the swing ($\varepsilon = 0$) is $\frac{2\pi}{\omega_o}$. Linearly stable regions in blue will in general bound unstable red regions (Arnold tongues). One expects instability to appear when we pump at $T$ values near $\frac{k\pi}{\omega_o}, k \in \mathbb{N}$. The standard way to pump a swing is with $T = \frac{2\pi}{\omega_o}$.

Exercises:

1. Let $u, v, c : [0,T] \to [0,\infty)$ with $u, v$ continuous and $c$ differentiable satisfying:
$$v(t) \le c(t) + \int_0^t u(s)v(s)\ ds, \quad t \in [0,T].$$
(a) Set $R(t) := \int_0^t u(s)v(s)\ ds$. Show that $\frac{dR}{dt} - uR \le uc$.
(b) Using an integrating factor, $\mu$, on part (a) show that $\frac{d}{dt}(\mu R) \le \mu uc$.
(c) Integrating the two sides of (b) from 0 to $t \le T$ deduce Gronwall's Lemma:
$$v(t) \le c(0) \exp\left(\int_0^t u(s)\ ds\right) + \int_0^t c'(s) \exp\left(\int_s^t u(\tau)\ d\tau\right)\ ds, \quad t \in [0,T]$$

2. Let $A, B$ be symmetric $n \times n$ matrices with $A$ positive definite. Show they may be simultanously diagonalized: there exists an invertible matrix $P$ such that $P^T AP = id, P^T BP = \Lambda$, with $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ and $\lambda_j \in \mathbb{R}$ the roots of $0 = \det(B - \lambda A)$.

3. Compute the inertia tensor (with respect to the origin) of a uniform (constant density) ellipsoid: $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$. Determine the principal axes and principal moments of inertia (eigenvectors/eigenvalues of this inertia tensor).

4. Let $B \subset \mathbb{R}^3$ be a solid region with center of mass at the origin, and $\mathbb{I}_o$ the inertia tensor of $B$ with respect to its center of mass. For $p \in \mathbb{R}^3$ with $\mathbb{I}_p$ the inertia tensor of $B$ with respect to $p$ and $\hat{\omega} \in \mathbb{R}^3$, a unit vector, show that [1]:
$$I_p = I_o + Md^2,$$
where $M$ is the total mass of $B$, $d$ the distance from $p$ to the line through the origin with direction $\hat{\omega}$, and $I_p := \mathbb{I}_p\hat{\omega} \cdot \hat{\omega}, I_o := \mathbb{I}_o\hat{\omega} \cdot \hat{\omega}$ are the moments of inertia of $\hat{\omega}$ wrt $p$ and the center of mass respectively.

5. Let $\omega, \nu \in \mathfrak{so}_3$ corresponding to $\vec{\omega}, \vec{\nu} \in \mathbb{R}^3$.
(a) Show that[2] $tr(\omega\nu) = -2\vec{\omega} \cdot \vec{\nu}$. In particular, one may write the kinetic energy of a rigid body in matrix or vector form as $K = \frac{1}{2}\mathbb{I}\vec{\omega} \cdot \vec{\omega} = -\frac{1}{4}tr(\omega\mathbb{I}\omega)$.
(b) Show that $[\omega, \nu] := \omega\nu - \nu\omega \in \mathfrak{so}_3$ corresponds to $\vec{\omega} \times \vec{\nu} \in \mathbb{R}^3$.
(c) Rewrite the matrix equations for the free rigid body ($\dot{C} = [C, \Omega]$, $C = \mathbb{I}_B\Omega$) in vector form:
$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \quad I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_1\Omega_3, \quad I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2$$
where $\vec{\Omega} = \Omega_1 e_1 + \Omega_2 e_2 + \Omega_3 e_3$ and $e_j$ the principal axes of $\mathbb{I}_B$ with moments of inertia $I_j > 0$.

6. Let $G = \mathrm{SO}_3$ and recall that $\mathfrak{g} = \mathfrak{so}_3$ was defined as the matrices $\xi = \frac{d}{dt}|_{t=0}g(t)$ for $g(t)$ a curve of rotations with $g(0) = e$ the identity (ie, $\mathfrak{g} = T_eG$ is the tangent space to $G$ at the identity).
(a) For $g \in G$, let $C_g : G \to G, h \mapsto ghg^{-1}$. Define the operator[3] $Ad_g = d_eC_g$ by $\mathfrak{g} \ni \xi = \frac{d}{dt}|_{t=0}g(t) \mapsto \frac{d}{dt}|_{t=0}C_g(g(t)) =: Ad_g(\xi)$. Show that $Ad_g : \mathfrak{g} \to \mathfrak{g}$.
(b) For $\xi = \frac{d}{dt}|_{t=0}g(t) \in \mathfrak{g}$, define the operator[4] $ad_\xi$ by $\mathfrak{g} \ni \eta \mapsto \frac{d}{dt}|_{t=0}Ad_{g(t)}\eta =: ad_\xi(\eta)$. Show that $ad_\xi : \mathfrak{g} \to \mathfrak{g}$.
(c) For $\xi, \eta \in \mathfrak{g}$, set $[\xi, \eta] := ad_\xi(\eta)$. Show that, for $G = \mathrm{SO}_3$, we have $[\xi, \eta] = \xi\eta - \eta\xi$.
(d) For $\xi, \eta \in \mathfrak{g}$, set $k(\xi, \eta) := tr(ad_\xi ad_\eta)$. Show that, for $G = \mathrm{SO}_3$, we have $k(\xi, \eta)$ is the Killing form of exercise # 5 (a) above.

[1] Called the *parallel axis theorem* or *Huygens-Steiner* theorem.
[2] The bilinear form $(\omega, \nu) \mapsto tr(\omega\nu)$ is called the *Killing form* on $\mathfrak{so}_3$.
[3] All together these are called the *adjoint representation* of $G$, $Ad : G \to \mathrm{GL}(\mathfrak{g}), g \mapsto Ad_g$. One has: $Ad_{gh} = Ad_g \circ Ad_h$ and $Ad_g([\xi, \eta]) = [Ad_g\xi, Ad_g\eta]$.
[4] All together these are called the *adjoint representation* of $\mathfrak{g}$, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \xi \mapsto ad_\xi$. One has: $ad_{[\xi,\eta]} = ad_\xi \circ ad_\eta - ad_\eta \circ ad_\xi$ (Jacobi identity).

7. Describe the behaviour of the linearized systems about the equilibrium points of:
   (a) the planar pendulum,
   (b) the spherical pendulum.

**§9 second variation**

Here we sketch some justifications to refer to the principle of *least* action, and related ideas.

First, we find a necessary condition for minimizers, analogous to 2nd derivative test. Let $q(t)$ be a trajectory of a Lagrangian mechanical system. For $t_o, t_1 \in \mathbb{R}$ and $q_\varepsilon(t)$ a fixed endpoint[1] variation of $q(t)$, we set $\eta(t) := \frac{d}{d\varepsilon}|_{\varepsilon=0} q_\varepsilon(t)$, and call

$$\delta^2 A_q(\eta,\eta) := \frac{d^2}{d\varepsilon^2}|_{\varepsilon=0} \int_{t_o}^{t_1} L(q_\varepsilon(t), \dot{q}_\varepsilon(t))\ dt$$

the *second variation* of $q(t)$. Clearly, if $q(t)$ is a minimizer among fixed endpoint curves from $q(t_o)$ to $q(t_1)$, then $\delta^2 A_q(\eta,\eta) \geq 0$ for any vector field $\eta(t)$ along $q(t)$ which vanishes at the endpoints. One has:

**Legendre condition:** If the Hessian $\partial_{\dot{q}}^2 L$ is positive definite, then for each extremal $q(t)$ and $t_o \in \mathbb{R}$ there exists $t_1 > t_o$ such that for any $t' \in (t_o, t_1)$ the trajectory $q(t)$ is a local[2] minimizer among curves with fixed endpoints $q(t_o), q(t')$.

*proof:* First we compute

$$(*) \qquad \delta^2 A_q(\eta,\eta) = \int_{t_o}^{t_1} P(t)\dot{\eta}\cdot\dot{\eta} + Q(t)\eta\cdot\eta\ dt$$

where $P(t) := \partial_{\dot{q}}^2 L(q(t),\dot{q}(t))$, $Q(t) := \left(\partial_q^2 L(q(t),\dot{q}(t)) - \frac{d}{dt}\partial_q\partial_{\dot{q}} L(q(t),\dot{q}(t))\right)^{sym}$. Consider the second order (linear) ode: $\frac{d}{dt}(P\dot{v}) = Qv$ with fundamental matrix solution

$$(**) \qquad \frac{d}{dt}(P\dot{V}) = QV.$$

For $V(t)$ an invertible solution of $(**)$, $WV = -P\dot{V}$ satisfies the (non-linear) first order matrix ode:

$$(***) \qquad Q + \dot{W} = WP^{-1}W.$$

Note that for a solution, $W(t)$, of $(***)$ so too is $W(t)^T$. Hence when $W(t_o)$ is symmetric, $W(t)$ is symmetric for all $t$ at which the solution is defined. For such solution $W(t)$, we have:

$$\begin{aligned}
\delta^2 A_q(\eta,\eta) &= \int_{t_o}^{t_1} P\dot{\eta}\cdot\dot{\eta} + Q\eta\cdot\eta\ dt + W\eta\cdot\eta|_{t_o}^{t_1} \\
&= \int_{t_o}^{t_1} P\dot{\eta}\cdot\dot{\eta} + 2W\eta\cdot\dot{\eta} + (Q+\dot{W})\eta\cdot\eta\ dt \\
&= \int_{t_o}^{t_1} P(\dot{\eta} + P^{-1}W\eta)\cdot(\dot{\eta} + P^{-1}W\eta)\ dt > 0
\end{aligned}$$

for all $\eta(t) \neq 0$ vanishing at the endpoints (since by assumption $P(t)$ is positive definite). Consider a solution of $(***)$ with $W(t_o) = id$, defined for some non-empty interval $[t_o, t_1)$ so that $\delta^2 A_q(\eta,\eta) > 0$ for every fixed endpoint variation $q(t_o), q(t')$ with $t' \in (t_o, t_1)$. One may further show that in an appropriate topology[3] on such fixed endpoint curves one has that $q(t)$ is a strict local minimum. $\square$

That is, extremals of natural mechanical systems are action minimizers *on sufficiently short time intervals.* The computations in the above proof lead to the important ode's $(**)$, $(***)$ called the *Jacobi equation* and *Riccati equation* respectively of the variational problem.

Observe that the Jacobi equation $(**)$ is the same as the linearized or variational equations along the trajectory $q(t)$, as well as the Euler-Lagrange equations for the functional $\eta \mapsto \int P\dot{\eta}\cdot\dot{\eta} + Q\eta\cdot\eta\ dt$.

[1] So $q_\varepsilon(t_o) = q(t_o), q_\varepsilon(t_1) = q(t_1)$.

[2] To make this precise, one needs to introduce a topology on the space of fixed endpoint curves.

[3] Namely, one may show that for any $\hat{q} \neq q$ with the same fixed endpoints as $q$ and $\sup_{t\in[t_o,t']} |\hat{q}(t) - q(t)| < \varepsilon$, $\sup_{t\in[t_o,t']} |\dot{\hat{q}}(t) - \dot{q}(t)| < 1$ with $\varepsilon$ sufficiently small that $A(\hat{q}) > A(q)$.

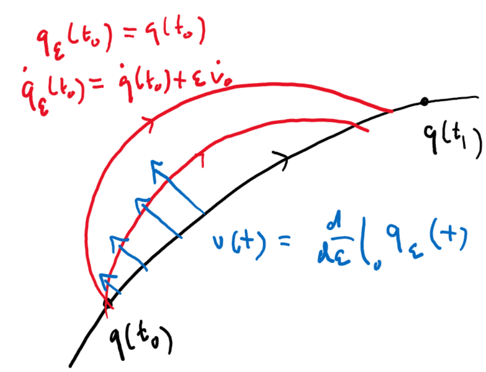


Figure 30. A *Jacobi field*, $v(t)$, is a vector field along a trajectory obtained as a variation of a 1-parameter family of trajectories. They are solutions of the variational equations: $\frac{d}{dt}(P\dot v) = Qv$. All Jacobi fields may be expressed in terms of the matrix solutions, $V_1, V_2$ of $(**)$, with initial conditions $V_1(t_o) = 0, \dot V_1(t_o) = I, V_2(t_o) = I, \dot V_2(t_o) = 0$, through $v(t) = V_1(t)\dot v_o + V_2(t)v_o$. The points $q(t_o)$ and $q(t')$ along a trajectory are said to be *conjugate points* if there exists a non-zero Jacobi field along $q(t)$ with $v(t_o) = v(t') = 0$. Geometrically, one pictures conjugate points as those for which there is a 1-parameter family of trajectories emanating from $q(t_o)$ and collecting at $q(t')$ to second order.

So the second variation relates to the variational equations (linearized equations around the trajectory). In particular, set $I(\eta,\xi) := \int_{t_o}^{t_1} P\dot\eta\cdot\dot\xi + Q\eta\cdot\xi \, dt$ for vector fields, $\eta, \xi$, along $q(t)$, called the *index form* of the trajectory, with $I(\eta) := I(\eta,\eta)$ the second variation, thought of as a quadratic form on vector fields along $q(t)$ (a vector space). The *Morse index* of $q(t)$ (from $q(t_o)$ to $q(t_1)$) is the dimension of the subspace of vector fields along $q(t)$ for which $I$ is negative definite. Then:

**Morse index:**[1] Suppose $q(t)$ is an extremal of a Lagrangian system satisfying the Legendre condition and defined on $[t_o, t_1]$ with $q(t_o), q(t_1)$ not conjugate points. Then the index of $q(t)$ is finite and equal to the number of interior conjugate points (counted with multiplicities) along $q(t)$, from $t_o$ to $t_1$.

*sketch:* The connection is made by considering $I(\eta)$ on the space of piecewise smooth vector fields along $q(t)$ (vanishing still at the endpoints). Suppose $\eta$, piecewise smooth on the subdivision $t_0 < s_1 < ... < s_k < t_1$, is an extremal of $\eta \mapsto I(\eta)$. Then integration by parts yields:

$$0 = \int_{t_o}^{t_1} \left(Q\eta - \frac{d}{dt}(P\dot\eta)\right)\cdot\delta\eta \, dt + \sum_1^k \left(\dot\eta(s_{j+}) - \dot\eta(s_{j-})\right)\cdot\delta\eta(s_j)$$

so that extremals are in particular smooth ($\dot\eta(s_{j+}) = \dot\eta(s_{j-})$) as well as being Jacobi fields. Now let $t' \in (t_o, t_1)$ be a conjugate point: there exists a non-zero Jacobi field $v(t)$ along $q(t)$ vanishing at $t_o$ and $t'$. Consider $\eta(t) = v(t)$ for $t \in [t_o, t']$ and $\eta(t) = 0$ for $t \in [t', t_1]$. Then since $\dot v(t') \neq 0$ (the Jacobi field is non-zero), $\eta(t)$ is not smooth (in particular not a minimizer). However one computes $I(\eta) = 0$, so that there must exist some vector field $\tilde v$ along $q(t)$ with $I(\tilde v) < 0$. In this way one has an association of conjugate points to vector fields along $q(t)$ along which $I$ is negative definite. Completing the proof requires showing that the entire span of the negative space of $I$ is obtained in this way. $\square$

Note that by the same computations and reasoning one obtains that the nullity of $I$ (dimension of space on which $I(\eta) = 0$) is equal to the mutiplicity of $t_o, t_1$ as conjugate points, ie number of independent Jacobi fields along $q(t)$ vanishing at $t_o, t_1$. In fact, applied to periodic orbits $q(t_o) = q(t_1)$, it follows that the multiplicity of $t_o, t_1$ as conjugate points is equal to the multiplicity of the eigenvalue 1 of the linearized Poincaré map. More precisely, letting $I(\eta,\xi) = \langle H\eta, \xi\rangle = \int_{t_o}^{t_1} H\eta\cdot\xi \, dt$, define $H$, the *Hessian of the second variation*, we have for periodic orbits a more complete relation to the Poincaré map via:

[1]See the excellent book *Morse theory* of Milnor, for the case of geodesics (free motions).

**Hill's formula:**[1] Let $q(0) = q(T)$ be a periodic orbit of a Lagrangian system on $\mathbb{R}^n$ (satisfying Legendre condition) with linearized Poincaré return map $P$ and Hessian $H$ of the second variation. Then:

$$\det(P - I) = (-1)^n \beta \det H$$

where $\beta > 0$ is a positive scaling constant.

- Hill's formula relates the Poincaré map to an infinite determinant[2]. Note that when $q(0)$ and $q(T)$ are conjugate points then both sides of the equation are just zero.

  When $q(0), q(T)$ are not conjugate, the sign of the left side is determined by the Morse index of the trajectory: $sign(\det(P - I)) = (-1)^{n+ind(H)}$.

  A first integral $F$ of a Lagrangian system leads in general to eigenvalues of 1 of the Poincaré map: for any $\xi \in \Sigma$, we have $F(P(q(0)+\xi)) = F(q(0)+\xi) \Rightarrow \nabla_{q(0)} F \cdot dP\xi = \nabla_{q(0)} F \cdot \xi$. So, for $v$ the projection of $\nabla_{q(0)} F$ to $\Sigma$, we have $dP^T v = v$, so $dP^T$ has an eigenvalue of 1, and hence as well $dP$.

  In particular, autonomous (time independent) Lagrangian systems have eigenvalues of 1 for Poincaré maps due to the energy integral. In this case, one works with a restricted system to a fixed energy level and considers variations having this fixed energy. One may apply a Hill formula then to this restricted system to obtain information on the eigenvalues of the restricted Poincaré map.

  For example (see Bolotin's survey for details), when the linearized Poincaré map at fixed energy, $P_E$, is non-degenerate (no eigenvalues of 1) one has

  $$sign(\det(P_E - I)) = (-1)^{n+ind(H)} sign(\frac{dE}{dT})$$

  where $dE/dT$ is the change in energy as a function of period of the 1-parameter family of periodic orbits (parametrized by energy).

  Considering say a geodesic flow on a surface, then $n = 2$ and changing the pure kinetic energy may be thought of as changing the constant speed of the parametrization. Hence in this case $\frac{dE}{dT} < 0$, and $sign(\det(P_E - I)) = (-1)^{ind(H)+1}$ where $ind(H)$ is the Morse index of the periodic geodesic. Also note that here $P_E$ is a $2 \times 2$ matrix (the flow takes place in the 4-dimensional tangent bundle to the surface, and a transverse to the periodic orbit intersected with fixed energy is 2 dimensional). Hence for a minimizing ($ind(H) = 0$) non-degenerate periodic trajectory, we have $sign(\det(P_E - I)) < 0$. Let $\chi(\lambda) = \det(P_E - \lambda I)$ be the characteristic polynomial (whose roots are eigenvalues). Then since $P_E$ is $2 \times 2$ we have $\chi(\lambda) \to \infty, \lambda \to \pm\infty$. In particular, since $\chi(1) < 0$, there must be an eigenvalue $\lambda > 1$, ie such minimizing periodic geodesics are unstable.

- We consider a periodic orbit $q(t + T) = q(t)$ of the Lagrangian system:

  $$L = \frac{|\dot{q}|^2}{2} + U(q), \quad q \in \mathbb{R}^2.$$

  The variational equations around $q(t)$ have the form:

  $$\dot{X} = \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Omega(t) & 0 \end{pmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = A(t)X$$

  where $\Omega(t) := \partial_q^2 U(q(t))$ is the (symmetric) Hessian matrix of $U$ evaluated along the trajectory. We are interested in the eigenvalues of the Poincaré map applied to solutions with the same fixed energy, $E = \frac{|\dot{q}|^2}{2} - U(q)$, as the given periodic orbit.

[1] See eg: S. Bolotin, D. Treschev. *Hill's formula.* Russian Mathematical Surveys 65.2 (2010): 191.

[2] Used by G. Hill in analysis of the motion of the moon: G. Hill, *Researches in the lunar theory.* American journal of Mathematics 1.1 (1878): 5-26. The convergence of these infinite determinants was examined by Poincaré: eg T. II, §. 185 of H. Poincar'e, *Les méthodes nouvelles de la mécanique céleste.* Gauthier-Villars, 1899.

Consider the fundamental matrix solution to the variational equations, $\dot{B} = A(t)B, B(0) = Id$, expressed in a basis $X_1, ..., X_4$ with $X_4 = (\dot{q}(0), \ddot{q}(0)), X_3 = \nabla E_{(q(0),\dot{q}(0))} = (-\ddot{q}(0), \dot{q}(0))$ and $X_1, X_2$ spanning $(X_3, X_4)^{\perp}$. Then $B(T)X_4 = X_4$ and $B(T)X_{1,2} \in span\{X_1, X_2, X_4\}$ by energy conservation. Hence:

$$B(T) = \begin{pmatrix} \hat{P} & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

where $\hat{P}$ is the linearized Poincaré map restricted to the fixed energy level. In particular, since $tr(A(t)) = 0$, we have $\det B = \det \hat{P} = 1$. In fact, it follows that for each $t$, the ode's satisfied by the solutions $X_1(t), X_2(t)$ of the variational equations with fixed energy may be written in an appropriate basis as a closed $2 \times 2$ traceless system:

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega(t) & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = a(t)x$$

with fundamental matrix $b(t)$ having $b(T) = \hat{P}$, ie a Hill equation $\ddot{\theta} = -\omega(t)\theta$, with $\omega(t+T) = \omega(t)$.

We will consider how one may arrive at Hill's formula for a special case of a Hill equation: the *Mathieu equation*:

$$\ddot{x} = -(a^2 + b\cos t)x.$$

Let $\rho = e^{2\pi i h}, \rho^{-1}$ be the eigenvalues of the Poincaré map and write the corresponding solution in Fourier series as:

$$x(t) = \sum \chi_n e^{(n+h)it}$$

Where the coefficients satisfy:

$$(*) \qquad \left(a^2 - (n+h)^2\right)\chi_n + \frac{b}{2}\left(\chi_{n+1} + \chi_{n-1}\right) = 0.$$

The fact that there exists a non-trivial solution to the infinite linear system $(*)$ leading to a convergent solution of the Mathieu equation corresponds to the vanishing of an infinite determinant:

$$\square(h) := \begin{vmatrix} \ddots & a_{-2} & 0 & 0 & 0 \\ a_{-1} & 1 & a_{-1} & 0 & 0 \\ 0 & a_0 & 1 & a_0 & 0 \\ 0 & 0 & a_1 & 1 & a_1 \\ 0 & 0 & 0 & a_2 & \ddots \end{vmatrix} = 0, \quad a_n(h) := \frac{b}{2(a^2 - (n+h)^2)}.$$

The roots of $\square(z) = 0$ are precisely $z = \pm(h+n)$, which coincide with the roots of $\cos 2\pi z - \cos 2\pi h$. Moreover, one may show that $\square(z)$ is an analytic function, and it follows that $\square(z) = A(\cos 2\pi z - \cos 2\pi h)$, for some $A$. Note that when $b = 0$, we have the explicit solution and $h = a$, so that *for* $b = 0$: $\square_o(z) = A_o(\cos 2\pi z - \cos 2\pi a) = 1$. Evaluating the quotients at $z = 0$, one obtains:

$$\frac{1 - \cos 2\pi h}{1 - \cos 2\pi a} = \square(0).$$

Observe that $\square(0)$ is exactly a rescaled determinant of the Hessian of the second variation, $Hx = \ddot{x} + (a^2 + b\cos t)x$, written in the Fourier basis $e^{int}$ and $1 - \cos 2\pi h = 1 - \frac{1}{2}tr(P) = \frac{1}{2}\det(P - I)$.

### §10 direct method

...see for example the notes here, based on these notes.

Exercises:

1. Consider a non-autonomous linear system of ode's: $\dot{X} = A(t)X, X \in \mathbb{R}^n$. Let $B(t)$ be a fundamental matrix solution: $\dot{B} = A(t)B$ of the system and $W(t) := \det B(t)$ the Wronskian. Show that
$$\dot{W} = tr(A(t))W.$$

2. Let $\mathrm{SL}_2(\mathbb{R})$ be the group of $2 \times 2$ matrices with determinant 1 and $\mathbb{RP}^1$ the set of lines through the origin in $\mathbb{R}^2$.

   (a) Show that $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{A \sim -A\}$ acts on $\mathbb{RP}^1$ by $span(v) \mapsto span(Av)$.

   (b) For $A(t) \in \mathrm{SL}_2(\mathbb{R})$ a smooth curve of matrices, show that $\dot{A}A^{-1}$ and $A^{-1}\dot{A}$ have trace zero.[1]

3. Let $A(t) \in \mathrm{SL}_2(\mathbb{R})$ be a smooth curve of matrices determining the plane curves $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A(t)\begin{pmatrix} x \\ y \end{pmatrix}$.

   (a) Show that these plane curves satisfy an ode of the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathfrak{a}(t)\begin{pmatrix} x \\ y \end{pmatrix}$ with $\mathfrak{a}(t) \in \mathfrak{sl}_2(\mathbb{R})$

   (b) For solutions of the linear ode in (a), show that $u := \frac{y}{x}$ satisfies a Riccati differential equation:
$$\dot{u} = a(t) + b(t)u + c(t)u^2.$$

4. Let $V, \langle\cdot,\cdot\rangle$ be an $n$-dimensional (real) inner product space, and $\beta : V \times V \to \mathbb{R}$ a symmetric bilinear with matrix representation $B : V \to V$ defined by $\beta(u, v) = \langle Bu, v\rangle$.

   Show that the eigenvalues of $B$ are critical values[2] of $V\backslash 0 \ni v \mapsto \frac{\beta(v,v)}{\langle v,v\rangle} \in \mathbb{R}$.

5. Consider the following boundary value problem: one seeks solutions $y(x)$ of the system
$$a(x)y'' + b(x)y' + c(x)y = f(x)$$
   with $(y(0), y'(0)) \in V_0, (y(1), y'(1)) \in V_1$, for $V_{0,1}$ given linear subspaces of $\mathbb{R}^2 \ni (y, y')$.

   (a) Show that there exists a function $w(x) > 0$ so that $w\,(ay'' + by' + cy) = (p(x)y')' + q(x)y$ is in 'Sturm-Liouville form' for some $p(x), q(x)$.

   (b) Consider the 'Sturm-Liouville' operator[3] $Hy := (py')' + qy$. For $\varphi_j(x)$ with $(\varphi_j(0), \varphi_j'(0)) \in V_0, (\varphi_j(1), \varphi_j'(1)) \in V_1$, show that:
$$\langle H\varphi_1, \varphi_2\rangle = \langle\varphi_1, H\varphi_2\rangle$$
   where $\langle H\varphi_j, \varphi_k\rangle := \int_0^1 (H\varphi_j)(x)\varphi_k(x)\ dx$.

6. Show that $y(x)$ with $y(0) = y(1) = 0$ satisfies $(py')' + qy = \lambda y$ for some $\lambda \in \mathbb{R}$ iff $y$ is an extremal of the functional $u \mapsto \frac{\int_0^1 p(u')^2 + qu^2\ dx}{\int_0^1 u^2\ dx}$ among $u(x) \not\equiv 0$ with $u(0) = u(1) = 0$.

7. Let $\Omega \subset \mathbb{R}^2$ be a compact convex region (a 'billiard table') with smooth boundary $\partial\Omega$. Let $\gamma(s) : [0, \ell] \to \partial\Omega$ be an arc-length parametrization (so $\gamma(0) = \gamma(\ell)$. Set $S(s_o, s_1) := |\gamma(s_o) - \gamma(s_1)|$.

   (a) Show that $\partial_{s_o}S = -\cos\varphi_o, \partial_{s_1}S = \cos\varphi_1$, where $\varphi_o, \varphi_1$ are the incidence angles of the boundary $\partial\Omega$ with the line from $\gamma(s_o)$ to $\gamma(s_1)$.

   (b) For $n \in \mathbb{N}$ Consider $A : [0, \ell]^{n+1} \to \mathbb{R}_{\geq 0}$ defined by:
$$A(s_o, s_1, ..., s_n) = S(s_o, s_1) + S(s_1, s_2) + ... + S(s_{n-1}, s_n) + S(s_n, s_o)$$
   Show that a a maximum of $A$ determines impact points of a periodic billiard orbit in $\Omega$.

[1] We write $\mathfrak{sl}_2(\mathbb{R})$ for the set of traceless $2 \times 2$ matrices.

[2] That is, for a function $f : \mathbb{R}^n \to \mathbb{R}$, its *critical points* are $v_o \in \mathbb{R}^n$ such that $d_{v_o}f = 0$ and $f(v_o) \in \mathbb{R}$ for $v_o$ a critical point is called a *critical value*.

[3] The relation of this computation to the previous exercise is that –analogous to linear algebra– one expects to be able to orthonormally diagonalize the operator $H$: $Hy_j = \lambda_j y_j$, so that the solution to the general boundary value problem $Hy = f(x)$ would be solved by $y = \sum \frac{c_j}{\lambda_j} y_j$ with $c_j = \int f y_j\ dx = \langle f, y_j\rangle$.

From a practical standpoint, the Hamiltonian formalism consists of rewriting the equations of motion, $m_j\ddot{q}_j = -\partial_{q_j}V$, in the more symmetric form of *Hamilton's equations*:

$$\dot{q}_j = \partial_{p_j}H, \quad \dot{p}_j = -\partial_{q_j}H$$

with $p_j = m_j\dot{q}_j$ the momenta, and $H = \sum\left(\frac{|p_j|^2}{2m_j}\right) + V(q)$ the energy or *Hamiltonian*.[1] At the moment it may be surprising that this rather unmotivated or 'naive' rewriting of the equations will lead to a great deal of insight as to the structure of the trajectories as well as new techniques.

To begin, we observe that Hamilton's equations are similar to a gradient flow, in fact for

$$X_H := -J\nabla H, \quad J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \nabla H = \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}$$

the *symplectic gradient* or *Hamiltonian vector field* of $H$, we have equations of motion:

$$\dot{x} = X_H, \quad x = \begin{pmatrix} q \\ p \end{pmatrix}.$$

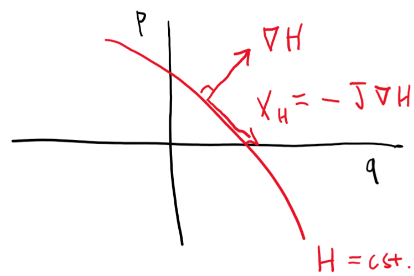


Figure 31. The symplectic gradient may be thought of as the usual gradient rotated by $\frac{\pi}{2}$ in each $q_j, p_j$ plane.

Defining the anti-symmetric bilinear form $\omega(\vec{u}, \vec{v}) := \vec{u}\cdot J\vec{v}$ –the *standard symplectic form* on $\mathbb{R}^{2n}$– we have:[2]

$$dH(*) = -\omega(X_H, *)$$

We remark that these equations are also nicely compatible under the identification with complex numbers where $\mathbb{R}^{2n} \ni x \leftrightarrow \vec{z} = (z_1, ..., z_n) = (q_1 + ip_1, ..., q_n + ip_n) \in \mathbb{C}^n$. Then the operator $J$ corresponds to $\vec{z} \mapsto i\vec{z}$ and $\omega(\vec{z}, \vec{w}) = \vec{z}\cdot i\vec{w}$ is the imaginary part of the standard Hermitian inner product:

$$\langle \vec{z}, \vec{w}\rangle = \sum z_j\bar{w}_j = \vec{z}\cdot\vec{w} + i\omega(\vec{z}, \vec{w}).$$

## §11 propagation: characteristics and geometric optics

We explain the 'optical analogy' [3] as a motivation for Hamilton's equations.

First, we recall a connection between first order ode's and first order pde's: the *method of characteristics*.[4] To begin, consider a first order implicit ode:

$$F(x, y, \frac{dy}{dx}) = 0.$$

[1]For a general Lagrangian system: $L(q, \dot{q})$, setting $p = \partial_{\dot{q}}L$ to implicitly define $\dot{q}(q, p)$ and with $H(q, p) := \dot{q}\cdot p - L$ the Euler-Lagrange equations take the same symmetric form of Hamilton's equations.

[2]Analogous to how usual gradient is defined with the dot product: $df(*) = (*)\cdot\nabla f$.

[3]Hamilton developed this formalism for mechanical problems from his work on optics: W. Hamilton, *Theory of systems of rays.* (1828).

[4]See Arnold's *Lectures on partial differential equations*, or *Geometrical methods in the theory of differential equations*

Geometrically, the solutions may be described as follows. First, $\{F = 0\} =: \Sigma$, defines a surface in the 'position and slope' space[1], $(x, y, p) \in \mathbb{R}^3$ with $p := \frac{dy}{dx}$ a slope at the point $(x, y) \in \mathbb{R}^2$. Every planar curve $(x, y(x)) \in \mathbb{R}^2$ lifts to a curve $(x, y(x), \frac{dy}{dx}(x)) \in \mathbb{R}^3$. Thus one seeks curves in the plane whose lifts to the position-slope space lie in the surface $\Sigma$.

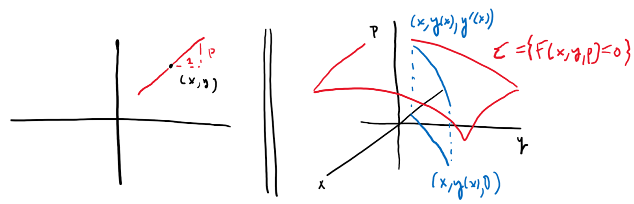


Figure 32. The set of 'planar contact elements' consists of pointed lines in the plane. It may be locally (omitting vertical lines) parametrized as $\mathbb{R}^3$ by position $(x, y)$ coordinates and a slope, $p$, coordinate. The first order ode defines a surface in $\mathbb{R}^3$, and one seeks plane curves whose lifts (point on curve lifts to point on curve and slope of tangent line to curve at the point) lie in this surface.

Not every curve of $\mathbb{R}^3 \ni (x, y, p)$ appears as the lift of a planar curve. Namely, the lifts of planar curves $(x(s), y(s))$ are characterized by satisfying the condition: $px' = y'$. Conversely, any curve $(x(s), y(s), p(s))$ satisfying $px' = y'$ projects to a planar curve $(x(s), y(s))$ of which it is the lift. This additional structure on $\mathbb{R}^3 \ni (x, y, p)$ is called a *contact structure* and geometrically consists of a field of 2-planes in $\mathbb{R}^3$: the two plane $\mathscr{D}_{(x,y,p)}$ is given by the span of tangent vectors to lifts of plane curves passing through $(x, y)$ with slope $p$ at $(x, y)$. Alternately, we may write:[2]

$$\mathscr{D} = \ker \alpha, \quad \alpha = pdx - dy.$$

Now, as for the implicit ode, its solutions may be described by considering the line field on $\Sigma$ given by intersecting the tangent space to the surface with the plane field $\mathscr{D}$. Integral curves of this line field on the surface project to solutions of the implicit ode.

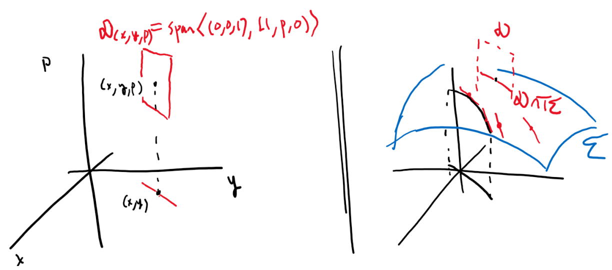


Figure 33. The plane curves whose lifts pass through a fixed point $(x, y, p)$ have tangents spanning a plane $\mathscr{D}_{(x,y,p)}$ (spanned by $(0, 0, 1), (1, p, 0)$). This distribution of 2-planes over $\mathbb{R}^3$ is the *standard contact structure*. By intersecting $\mathscr{D}$ with the tangent spaces to $\Sigma$ one obtains a line field on $\Sigma$, integral curves of which project to solutions of the ode.

This geometric picture generalizes to a similar description of solutions to first order pde's. Let us first consider some simpler cases before outlining the general scheme.

Solutions to a linear first order pde of the form

$$v_1(x)\partial_{x_1} y + ... + v_n(x)\partial_{x_n} y = 0$$

where $x \in \mathbb{R}^n, y : \mathbb{R}^n \to \mathbb{R}$, are equivalent to first integrals of the first order ode $\dot{x} = v(x)$. The level sets of $y$ thus consist of unions of trajectories of $\dot{x} = v(x)$. Solutions to the Cauchy problem, $y|_\Gamma = y_o$, where $\Gamma \subset \mathbb{R}^n$ is a hypersurface and $y_o : \Gamma \to \mathbb{R}$ a given initial condition are thus generated by propagating the initial values of $y_o$ along the trajectories of $\dot{x} = v(x)$.

[1]One could also call this the 'space of pointed lines in the plane' or *space of planar contact elements*.

[2]The notation $\alpha = pdx - dy$ may be understood as $\alpha : \mathbb{R}^3 \to \mathbb{R}$, by $\alpha(\dot{x}, \dot{y}, \dot{p}) = p\dot{x} - \dot{y}$, ie $\alpha$ is a '1-form' on $\mathbb{R}^3$.

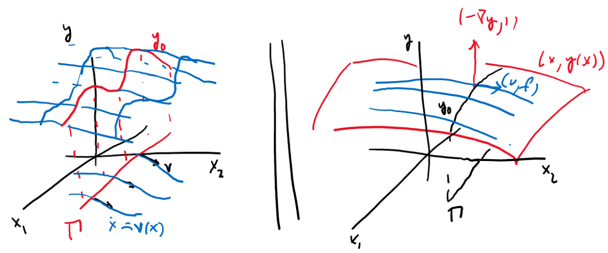


Figure 34. Graphs of solutions to linear (or quasi-linear) first order pde's consist of 'unions of characteristics'. That is integral curves of $\dot{x} = v(x)$ (or $\dot{x} = v(x, y), \dot{y} = f$).

Similarly, for a quasi-linear first order pde of the form:

$$v_1(x, y)\partial_{x_1} y + ... + v_n(x, y)\partial_{x_n} y = f(x, y)$$

the solutions graphs $(x, y(x)) \subset \mathbb{R}^n \times \mathbb{R}$ have normal $(-\nabla y, 1)$ perpendicular to the vector field $(v_1, ..., v_n, f)$. Thus solutions to the Cauchy problem, $y|_\Gamma = y_o$, may be generated by propagating the initial values of $y_o$ along the trajectories of $\dot{x} = v(x, y), \dot{y} = f(x, y)$.

Example:

- Consider the *inviscid Burgers' equation*: $u_t + uu_x = 0$ with initial condition $u(x, 0) = f(x)$. The characteristics are the integral curves of $\dot{t} = 1, \dot{x} = u, \dot{u} = 0$. Those with initial condition complying with the initial data, $(x_o, 0, f(x_o))$, are parametrized by $(f(x_o)t + x_o, t, f(x_o))$. When the parametrized surface: $(s, t) \mapsto (f(s)t + s, t, f(s))$ is a graph, this is a solution (however this surface is in general only locally a graph). This may be seen more easily by taking 'snaphsots' at fixed $t$ values.

  For each *fixed* $t$ one may see the evolution of the graphs $x \mapsto u(x, t)$, through the evolution of the plane curves parametrized by $s \mapsto (f(s)t + s, f(s))$. In general, these plane curves will cease to be graphs at some time by becoming 'vertical'. For example, with initial data $f(x) = x$, we may eliminate $x_o$ to obtain $u(x, t) = \frac{x}{1+t}$, which for $t = -1$ become 'vertical' ceasing to be graphs.

In the general case of an implicit first order pde:

$$F(x, y, \nabla y) = 0$$

one views $\{F = 0\} = \Sigma^{2n}$ as defining a hypersurface in $\mathbb{R}^{2n+1} \ni (x, y, p)$ with $p = (p_1, ..., p_n) = (\partial_{x_1} y, ..., \partial_{x_n} y) = \nabla y$, and one seeks graphs $(x, y(x))$ whose lifts, $p(x) = \nabla y(x)$, to $\mathbb{R}^{2n+1}$ lie in this hypersurface. As in the previous two cases, such lifts may be (locally) generated as unions of trajectories of a certain vector field on $\Sigma$, ie 'initial data propagates along characteristics'.

The construction of this vector field proceeds as follows. First, every lift of a graph $(x, y(x))$, leads to an $n$-dimensional subset of $\mathbb{R}^{2n+1}$ with tangent planes contained in the contact distribution of $2n$-dimensional hyperplanes:

$$\mathscr{D} := \ker \alpha, \quad \alpha = p \cdot dx - dy.$$

The intersection of $\mathscr{D}$ with the tangent spaces to $\Sigma$ leads to a field of $2n - 1$ dimensional planes, $\pi$, along $\Sigma$. A solutions graph may thus be obtained as the projection, $(x, y, p) \mapsto (x, y)$, of an $n$-dimensional subset of $\Sigma$ whose tangent spaces are contained in $\pi$. Consider such an $n$-dimensional subset, $\sigma^n$, of $\Sigma$. Then, by the generalized stokes theorem, one has:

$$0 = \int_{\partial D} \alpha = \int_D d\alpha$$

for *any* two dimensional region $D \subset \sigma$ with boundary $\partial D$. In particular, $d\alpha|_{T\sigma} \equiv 0$, where $d\alpha$ is the skew-symmetric bilinear form:[1] $dp \wedge dx$ on $\mathbb{R}^{2n+1}$. In general, $d\alpha|_{T\Sigma}$ is a non-degenerate skew symmetric

[1] This is another notation for the standard symplectic form on $\mathbb{R}^{2n} \ni (x, p)$, thought of as a (degenerate) bilinear form on $\mathbb{R}^{2n+1} \ni (x, y, p)$ by ignoring the $y$ components of tangent vectors.

bilinear form on $T\Sigma$ and may be used to define 'orthogonal complements'[1]. In particular, the orthogonal complements, $\pi^\perp \subset T\Sigma$, of the $2n-1$ dimensional planes $\pi$ give a line field on $\Sigma$ *contained in* $\pi$. In fact, provided $d\alpha|_{T\Sigma}$ is non-degenerate, one has that $\pi^\perp \subset T\sigma$, since otherwise $T\sigma + \pi^\perp$ is an $n+1$ dimensional subspace with orthogonal complement of dimension at least $n+1$. Hence, the surfaces we seek may be formed by propagating initial data along the integral curves of the line field $\pi^\perp$ on $\Sigma$. Explicitly, one may work out that these 'characteristics' are given by:

$$\dot{x} = \partial_p F, \quad \dot{y} = p \cdot \partial_p F, \quad \dot{p} = -(\partial_x F + F_y p).$$

We now consider propagation in *geometric optics*: light in a homogeneous medium is thought of as propagating at constant speed along straight lines, the 'light rays' or 'lines of sight' to an object.

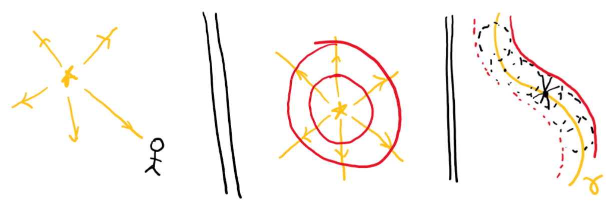


Figure 35. Light propagates with constant speed along straight lines. The 'line of sight' reaching us as an observer from a light source may be characterized as that which takes the least time. A point light source produces wave fronts (circles), thought of as where the light emitted from the source has reached after a given time. For a more complicated source, say a plane curve each of whose points are light sources, the wave front may be characterized by *Huygen's principle*: the wave front is the *envelope* of the wave fronts from each point on the curve (a curve tangent to all the circles of fixed radius with centers on the curve).

Light propagation from a source may be visualized in two equivalent ways. First, the light rays themselves from a source at position $x$ to an observer at point $q$ may be characterized variationally by *Fermat's principle of least time*: the path along which light travels from $x$ to $q$ is that for which its time of travel is minimized. Since here we consider the velocity of light to be constant (homogeneous medium) this is equivalent to the light traveling along a path from $x$ to $q$ which minimizes distance, ie straight line from $x$ to $q$. Second, the propagation of light from a source may be visualized with *wave fronts*: the boundary of the set where the light from the source may reach in time at most $t$.

In the plane, wlog with the velocity of light $c = 1$, consider a plane curve $\gamma$ as the light source. The wave fronts are then level sets of:

$$S(q) = dist(q, \gamma)$$

since $S(q) = t = cst.$ consists of points reachable from $\gamma$ in time $t$.

Now we consider the rays of light emitted from $q \in \gamma$ which reach the wave front at time $t$. Since the distance from $S^{-1}(t)$ to $\gamma$ is $t$, such light rays are perpendicular to the wave fronts and $\gamma$, ie they are the light rays normal to $\gamma$. Thus the 'light particle' moving by $t \mapsto q(t) = q + tn$ where $q \in \gamma$ and $n$ is normal to $\gamma$ at $q$, lies in the wave fronts for each[2] $t$. Observe that $S(q(t)) = t$, so that $\nabla_{q(t)} S \cdot \dot{q}(t) = 1$. Moreover, $\nabla_{q(t)} S$ and $\dot{q}(t)$ are both perpendicular to the wave front at time $t$. Hence

$$\nabla_q S = \dot{q}.$$

In particular since the velocity of light is unit, we have that $S$ satisfies the *eikonal equation*:

$$|\nabla S| = 1.$$

Thus propagation according to uniform motion of light particles along straight lines – solutions of the ode $\ddot{q} = 0, |\dot{q}| = 1$ – is along wave fronts: level sets of solutions of the pde $|\nabla S| = 1$.

[1]For $V$ a subpsace of $T\Sigma$, then $V^\perp$ is the subspace of $T\Sigma$ spanned by vectors, $w \in T\Sigma$, satisfying $d\alpha(v, w) = 0$, $\forall v \in V$. Note that because $d\alpha|_{T\Sigma}$ is skew-symmetric, orthogonal complements need not be complementary subspaces, ie one may have $V^\perp \cap V \neq 0$. However, if $d\alpha|_{T\Sigma}$ is non-degenerate, then the dimensions of complementary subspaces always sum to $2n = dim(T\Sigma)$.

[2]Strictly speaking this may in general only be true for sufficiently small $t$ values.

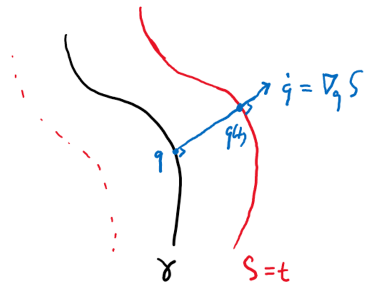


Figure 36. The gradient of a solution to the eikonal equation gives velocities of the light particles reaching the wave front.

Now, we make a connection to the the method of characteristics with pde's: we consider the eikonal equation defining the hypersurface $\Sigma = \{S_x^2 + S_y^2 = 1\} \subset \mathbb{R}^5 \ni (x, y, S, S_x, S_y)$ with contact form $\alpha = S_x dx + S_y dy - dS$. We may parametrize $\Sigma$ as $\mathbb{R}^3 \times S^1 \ni (x, y, S, \theta)$ where $(S_x, S_y) = (\cos\theta, \sin\theta)$. In this case, $d\alpha|_{T\Sigma}$ is degenerate [1], however we have on $\mathbb{R}^2 \times S^1 \ni (x, y, \theta)$ that $\pi$ projects at each point to span the whole space and $\pi^\perp$ projects to the line field spanned by $R := (\cos\theta, \sin\theta, 0)$ with characteristics:

$$(*) \quad \dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \quad \dot{\theta} = 0.$$

Note that for $\bar{\alpha} := \cos\theta\ dx + \sin\theta\ dy$, that $d\bar{\alpha}(R, *) \equiv 0$ determines $R$ upto scale. Moreover, by definition, a solution to the eikonal equation has:

$$dS = \bar{\alpha}$$

over its 'graph', $(x, y, \theta(x, y))$ where $\theta(x, y)$ is the angle of $\nabla_{(x,y)} S$. In particular, the vector field $\nabla S = (\cos\theta, \sin\theta)$ is the projection of $R$ to the plane.

Now, observe that the space $\mathbb{R}^2 \times S^1$ may be thought of as the space of 'oriented contact elements': $\xi = (x, y, \theta) \in \mathbb{R}^2 \times S^1$, where $\theta$ is the angle of a unit vector from the $x$-axis based at the point $(x, y)$. Here in place of the 2'nd order ode, $\ddot{q} = 0, |\dot{q}| = 1$, on the plane we have the first order ode $(*)$ above describing propagation of light particles with constant speed along rays. Plane curves, $\gamma(s)$, may be lifted by taking $\theta(s)$ as the angle of a unit normal to $\gamma$ at $\gamma(s)$. This generates a contact structure on $\mathbb{R}^2 \times S^1$ given by:

$$\mathscr{D} = \ker\bar{\alpha}$$

with $\bar{\alpha} = \cos\theta\ dx + \sin\theta\ dy$ as above.

In summary, we have the following structure in geometric optics:

- the variational 'least time' principle and light particle trajectories: $\ddot{q} = 0, |\dot{q}| = 1$,
- wave fronts: level sets of $S(q) = dist(q, \gamma)$ and the eikonal equation, $|\nabla S| = 1$,
- am 'extended space', $\mathbb{R}^2 \times S^1$ with a 1-form $\bar{\alpha} = \cos\theta\ dx + \sin\theta\ dy$. Over graphs of solutions to the eikonal equation we have $\bar{\alpha} = dS$,
- a vector field, $R$, on $\mathbb{R}^2 \times S^1$, defined by $d\bar{\alpha}(R, *) \equiv 0, \bar{\alpha}(R) = 1$, whose projected integral curves are light particle trajectories,
- solutions to the eikonal equation may be obtained from propagating initial data along trajectories $\dot{\xi} = R$,
- trajectories of $\dot{\xi} = R$ are contained in graphs of solutions to the eikonal equation coming from lifts of curves satisfying $\dot{q} = \nabla_q S$.

Upon replacing the least time variational principle by our 'least action' variational principle of mechanical systems, we will find analogous structure – the structure of the Hamiltonian formalism.

[1] One has $\pi^\perp = span(\cos\theta\partial_x + \sin\theta\partial_y, \partial_S)$

## §12 Hamilton-Jacobi equation

We consider a Lagrangian system given by an autonomous (time independent) Lagrangian: a function $L(q, v)$ of positions and velocities.

The variational principle characterizes trajectories with fixed endpoints over fixed time intervals. For $\mathbb{R}^n \times \mathbb{R} \ni (q, t)$ the *extended configuration space* of positions and times, a trajectory $q(t)$ naturally lifts to a trajectory in the extended space as a graph: $(q(t), t)$. Fix a point $(q_o, t_o)$ and suppose that for each $(q, t)$ we have selected a trajectory $(\gamma_{q,t}(\tau), \tau)$ going from $(q_o, t_o)$ to $(q, t)$ in such a way that the function:

$$S(q,t) := A(\gamma_{q,t}) = \int_{t_o}^{t} L(\gamma_{q,t}(\tau), \dot{\gamma}_{q,t}(\tau))\, d\tau$$

is differentiable.

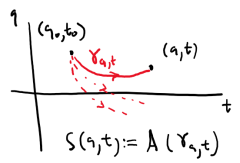


Figure 37. For example, when the Legendre condition is satisfied –as it is for mechanical systems– sufficiently short trajectories emanating from $q_o$ are minimizers, and one may take $S(q,t) := \min_{\gamma(t_o)=q_o, \gamma(t)=q} A(\gamma)$. This function $S(q,t)$ –*Hamilton's principal function*– is analogous to the 'wave front' function, $S(q) = dist(q, q_o)$, we met in geometric optics for a point source of light. When the Legendre condition holds, the locally defined $S(q,t) = \min_{\gamma(t_o)=q_o, \gamma(t)=q} A(\gamma) = A(\gamma_{q,t})$, may be extended along these locally minimizing trajectories to obtain an $S(q,t)$ which is differentiable whenever $(q_o, t_o), (q, t)$ are not conjugate points along $\gamma_{q,t}$. By the Morse index theorem the set of points where such an $S(q,t)$ fails to be differentiable is a discrete set.

Now, for $\eta \in \mathbb{R}^n$, we compute:

$$\partial_q S(q,t) \cdot \eta = d_{(q,t)} S(\eta, 0) = \frac{d}{d\varepsilon}|_{\varepsilon=0} S(q + \varepsilon\eta, t) = \int_{t_o}^{t} \partial_q L \cdot \eta + \partial_v L \cdot \dot{\eta}\, d\tau = \partial_v L(q(t), \dot{q}(t)) \cdot \eta$$

where $\dot{q}(t) := \dot{\gamma}_{q,t}(t)$ is the velocity of the extremal, $\gamma_{q,t}$, from $q_o$ to $q$ at time $t$. Or, for short:

$$\partial_q S = \partial_v L =: p.$$

Similarly:

$$\partial_q S(q,t) \cdot \dot{q}(t) + \partial_t S(q,t) = d_{(q,t)} S(\dot{q}(t), 1) = \frac{d}{d\varepsilon}|_{\varepsilon=0} S(\gamma_{q,t+\varepsilon}, t + \varepsilon) = L(q(t), \dot{q}(t))$$

$$\Rightarrow \partial_t S = L - p \cdot \dot{q} =: -H.$$

In summary, we have the equality:

$$dS = p \cdot dq - H dt,$$

over points $(q, t) = (\gamma_{q,t}(t), t)$ with $\gamma_{q,t}$ the chosen trajectory from $q_o$ to $q$.

In the derivations above we come across the *Legendre transform*, which may be defined solely in terms of the Lagrangian $L(q, v)$. Namely, for fixed $q$ we consider:

$$p := \partial_v L(q, \dot{q})$$

to define $\dot{q}(q, p)$ implicitly, calling $p$ the *momentum* at $q$ corresponding to the velocity $\dot{q}$ at $q$. Next, the *Hamiltonian*, or energy, is the function of position and momentum:

$$H(q, p) := p \cdot \dot{q}(q, p) - L(q, \dot{q}(q, p)).$$

Thus, from the *phase space* –pairs $(q, p) \in \mathbb{R}^{2n}$ of positions and momenta– we have the extended phase space $\mathbb{R}^{2n} \times \mathbb{R} \ni (q, p, t)$ with 1-form[1]:

$$\alpha := p \cdot dq - H dt.$$

The function $S(q, t)$ above satisfies –by definition of $p, H(q, p)$– the *Hamilton-Jacobi equation*:

$$H(q, \partial_q S) = -\partial_t S,$$

which is the mechanical analogue of the eikonal equation in geometric optics.

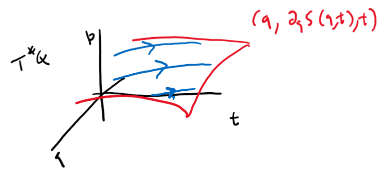


Figure 38. Graphs in the extended phase space of solutions to the Hamilton-Jacobi equation are invariant: that is they contain graphs of trajectories $(q(t), p(t), t)$ of Hamilton's equations: $\dot{q} = \partial_p H, \dot{p} = -\partial_q H$. When for example $\partial_q S(q, t) = \partial_q S(q)$ is independent of $t$, one obtains an invariant set, $(q, \partial_q S(q))$, in the phase space $\mathbb{R}^{2n} \ni (q, p)$.

Now, one may show that the mechanical system has the following structure analogous to what we found in optics:

- the variational principle, $q(t) \mapsto \int L(q, \dot{q})\, dt$, and the Euler-Lagrange equations.
- Hamilton's principal functions: $S(q, t) = \min_{\gamma(t_o)=q_o, \gamma(t)=q} A(\gamma)$ and the Hamilton-Jacobi equation, $H(q, \partial_q S) = -\partial_t S$.
- the extended phase space $\mathbb{R}^{2n} \times \mathbb{R} \ni (p, q, t)$ with Poincaré-Cartan 1-form $\alpha = p \cdot dq - H dt$. Over 'graphs', $(q, \partial_q S(q, t), t)$, of solutions to the Hamilton-Jacobi equation one has $dS = \alpha$.
- a vector field, $R$, on $\mathbb{R}^{2n} \times \mathbb{R}$ determined by $d\alpha(R, *) \equiv 0$ and $t' = 1$, whose projected integral curves to $\mathbb{R}^n \ni q$ are solutions of the Euler-Lagrange equations. Moreover, $R = (X_H, 1)$ where $dH(*) = -\omega(X_H, *)$ determines the Hamiltonian vector field, $X_H$, of $H$.
- Solutions to the Hamilton-Jacobi equation may be obtained by propagating initial data along trajectories of $R$.
- Trajectories of $R$ are contained in graphs of solutions to the Hamilton-Jacobi equation. These curves are lifts of solutions to the first order ode $\partial_v L(q, \dot{q}) = \partial_q S(q, t)$.

Let us postpone the proofs for later (see end of this section). For the moment, let us explain how the Hamiltonian formalism leads to a new approach for describing trajectories of a mechanical system.

Note that the equations of motion are determined entirely by $H$ and the standard symplectic form $\omega$ on $\mathbb{R}^{2n}$. A scheme to understand the trajectories may thus be to seek a transformation, $(q, p) \overset{\varphi}{\mapsto} (Q(q, p), P(q, p)) \in \mathbb{R}^{2n}$ which preserves[2] $\omega$ –called a *symplectic transformation*– and puts $H$ into a 'simple form'. Namely some form $H(Q, P)$ for which we can solve Hamilton's equations. If this can be done, we may then apply $\varphi^{-1}$ to these $(Q, P)$ trajectories to describe the solutions in the original, $(q, p)$, coordinates.

For example, if $(q, p) \overset{\varphi}{\mapsto} (Q(q, p), P(q, p))$ preserves $\omega$ and we have $H(Q, P) = P_1$ then the equations of motion are simply:

$$\dot{Q}_1 = 1, \quad \dot{Q}_j = 0 \ \ (j \neq 1), \quad \dot{P}_j = 0$$

[1]Called the Poincaré-Cartan form.

[2]Meaning '$dP \wedge dQ = dp \wedge dq$' or more explicitely: $\omega(d\varphi\, \vec{v}, d\varphi\, \vec{u}) = \omega(\vec{v}, \vec{u})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^{2n}$.

and the original solutions are $(q(t), p(t)) = \varphi^{-1}(t + t_o, Q_2^o, ..., Q_n^o, P_1^o, ..., P_n^0)$ for $t_o, Q_2^0, ..., Q_n^o, P_j^o$ constants. Thus the problem of solving the equations of motion is converted to the problem of finding a symplectic transformation for which $H$ takes on a manageable form.

Example:

- Consider linear oscillations $\ddot{x} = -x,\ x \in \mathbb{R}$. The Lagrangian is $L = \frac{\dot{x}^2 - x^2}{2}$ with Hamiltonian $H = \frac{p_x^2 + x^2}{2}$, $p_x = \dot{x}$ and $\omega = dp_x \wedge dx$ the standard area form on the $(p_x, x)$ plane. Set $I := \frac{p_x^2+x^2}{2} = \frac{1}{2}r^2$ with $p_x = r\cos\theta, x = r\sin\theta$. Then $dp_x \wedge dx = rdr \wedge d\theta = dI \wedge d\theta$. So that $(x, y) \mapsto (I, \theta)$ is a symplectic transformation in which $H = I$. Hence $\dot{I} = 0, \dot{\theta} = 1$ and the solutions $I(t) = I_o, \theta(t) = \theta_o + t$ give the solutions $p_x(t) = \sqrt{2I_o}\cos(\theta_o + t)$, $\ x(t) = \sqrt{2I_o}\sin(\theta_o + t)$ with $I_o, \theta_o$ constants.

Figure 39. In place of solving the original ode's, one can seek coordinate changes in which the ode's take a simple form.

One would thus like a manner in which to determine symplectic transformations. First:

*The flow by the symplectic gradient, $X_F$, of a function $F : \mathbb{R}^{2n} \to \mathbb{R}$, is a symplectic transformation.*

In fact, by defining a *symmetry* of $H$ to be a family of symplectic transformations $\varphi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, we obtain:

**Noether's theorem v3:** The Hamiltonian system, $H : \mathbb{R}^{2n} \to \mathbb{R}$, admits a symmetry iff it admits a first integral.

*proof:* First, suppose $H$ has a first integral, $F : \mathbb{R}^{2n} \to \mathbb{R}$. Then $0 = dF(X_H) = -\omega(X_F, X_H) = \omega(X_H, X_F) = -dH(X_F)$. That is $H$ is a first integral of $F$, so that $H$ is preserved by the (symplectic) flow of $X_F$, ie the flow of $X_F$ is a symmetry of $H$. Conversely, if $\varphi_t$ are symmetries of $H$, with $X(x) := \frac{d}{dt}|_{t=0}\varphi_t(x)$, one has that $0 = d(\omega(X, \cdot)) \Rightarrow -\omega(X, \cdot) = dF$ for some $F : \mathbb{R}^{2n} \to \mathbb{R}$. Hence $X = X_F$ is a symplectic gradient and: $0 = dH(X_F) = -dF(X_H)$, ie $F$ is a first integral of $X_H$. $\square$

Symplectic transformations may also be given by the use of *generating functions*: for $S(q, Q) : \mathbb{R}^{2n} \to \mathbb{R}$,

$$dS = \partial_q S \cdot dq + \partial_Q S \cdot dQ$$

so that by setting:

$$p := \partial_q S(q, Q), \quad P := -\partial_Q S(q, Q)$$

we have $0 = d(dS) = dp \wedge dq - dP \wedge dQ$. Now, *if* $p = \partial_q S(q, Q)$ is invertible to implicitly determine $Q(q, p)$ one then has $P(q, p) = -\partial_Q S(q, Q(q, p))$, and so a symplectic transformation $(q, p) \mapsto (Q(q, p), P(q, p))$.

A general solution to the Hamilton-Jacobi equation of the form:

$$H(q, \partial_q S(q, P)) = h(P)$$

depending on $n$ parameters $P_1, ..., P_n$ may then lead to a generating function and symplectic coordinates defined implicitly through $p = \partial_q S$, $\ Q = \partial_P S$ in which the trajectories are simply:

$$Q = Q_o + t\ h'(P_o), \quad P = P_o = cst..$$

Thus one new approach to solving a mechanical system is to seek a general solution depending on $n$-parameters to a Hamilton-Jacobi equation. Solving this first order pde is sometimes more straightforward

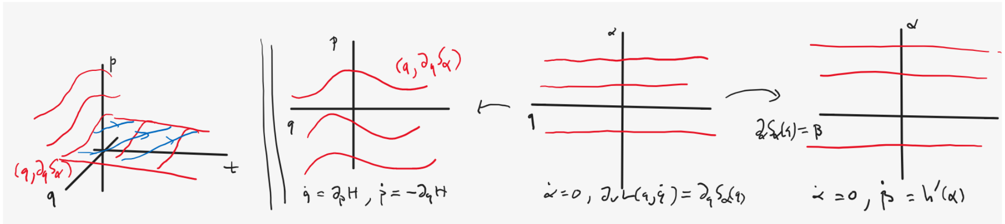


Figure 40. Applying seperation of variables $\tilde{S}(q,t) = S(q) + S_1(t)$ to the Hamilton-Jacobi equation leads to the time-independent or 'fixed energy' Hamilton-Jacobi equation: $H(q, \partial_q S(q)) = E$. Such solutions lead to invariant sets $(q, \partial_q S) \in \mathbb{R}^{2n}$ lying in the energy level $E$ and whose trajectories are determined as solutions to the first order ode $\partial_v L(q, \dot q) = \partial_q S(q)$. Now, suppose one has a family of such solutions, $H(q, \partial_q S_\alpha(q)) = h(\alpha)$, depending on parameters $\alpha$, in such a way that $(q, \alpha) \mapsto (q, \partial_q S_\alpha) = (q, p)$ is invertible. Then the parameters $\alpha(q,p)$ are first integrals of the system. By chain rule, $h'(\alpha) = \partial_p H \cdot \partial_\alpha \partial_q S_\alpha = \partial_q \partial_\alpha S_\alpha \cdot \dot q = \dot \beta$ where $\beta := \partial_\alpha S_\alpha(q)$. If as well $(q, \alpha) \mapsto (\alpha, \partial_\alpha S_\alpha) = (\alpha, \beta)$ is invertible then the system is solved. For this reason, one says that a general solution to the Hamilton-Jacobi equation depends on $n$-parameters.

to carry out than solving the system of first order ode's (Hamilton's equations). For instance, one may in some cases apply separation of variables to solve the Hamilton-Jacobi equation. In this way (see §13) Jacobi re-solved and 'explained' the integrability of the following mechanical problems:

- a point mass in the plane attracted by two fixed points according to Newton's inverse square law,
- the motion of a free particle (geodesic) on a quadratic surface (eg an ellipsoid).

Finally, let us explain some geometric descriptions of the Legendre transformation. In the projective plane, $\mathbb{P}^2$, there is a duality between points and lines. A *point* of the projective plane is a line in $\mathbb{R}^3$ through the origin and a *line* in the projective plane is a plane in $\mathbb{R}^3$ passing through the origin. The set of lines in $\mathbb{P}^2$ is called the *dual projective plane*, denoted $\mathbb{P}^{2*}$.

*Projective duality* is related to the incidence relations between points of $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. Namely for $p \in \mathbb{P}^2$ and $\ell \in \mathbb{P}^{2*}$ one may have either $p \in \ell$ or $p \notin \ell$, since $\ell \subset \mathbb{P}^2$ is a line in the projective plane. By definition, a point $\ell \in \mathbb{P}^{2*}$ determines a subset of $\mathbb{P}^2$ –namely the line in the projective plane. Conversely a point $p \in \mathbb{P}^2$ determines a subset (a 'line' in $\mathbb{P}^{2*}$ or 'pencil of lines') of $\mathbb{P}^{2*}$ consisting of all the lines passing through $p$:

$$p \mapsto p^* = \{\ell \in \mathbb{P}^{2*} : p \in \ell\} \subset \mathbb{P}^{2*}.$$

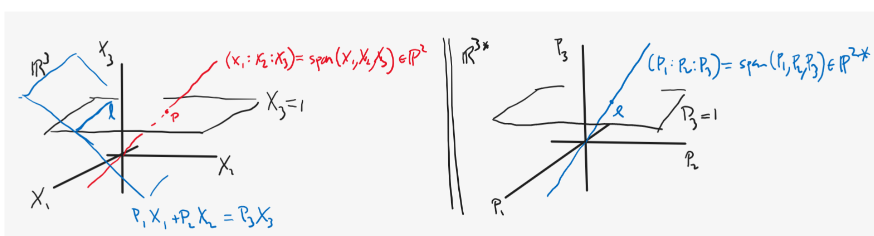


Figure 41. The projective plane is the set of lines through the origin in $\mathbb{R}^3$, $(X_1 : X_2 : X_3) = span(X_1, X_2, X_3) \in \mathbb{P}^2$. One may parametrize certain points of $\mathbb{P}^2$ as points of the usual plane via an *affine chart*, eg by intersections with the plane $X_3 = 1$. The dual projective plane is the set of planes through the origin in $\mathbb{R}^3$. It is a projective plane itself: the set of lines through the origin in $\mathbb{R}^{3*}$, where $\mathbb{R}^{3*}$ are the linear functionals $\mathbb{R}^3 \to \mathbb{R}$. A (non-zero) linear functional $\alpha : \mathbb{R}^3 \to \mathbb{R}$ determines the plane $\ker \alpha \subset \mathbb{R}^3$ in $\mathbb{P}^{2*}$.

Given a curve, $\gamma \subset \mathbb{P}^2$, in the projective plane there is a corresponding *dual curve* $\gamma^* \subset \mathbb{P}^{2*}$ consisting of the tangent lines to $\gamma$. Then, in general, $(\gamma^*)^* = \gamma$. Also, this may all be done for a general dimension, $\mathbb{P}^n$.

Now, the Legendre transform[1] of $L(q, v)$ may be described as follows. For fixed $q$, we consider the graph

[1]Here we consider that $L(q, v)$ is convex in $v$, as is the case for mechanical systems.

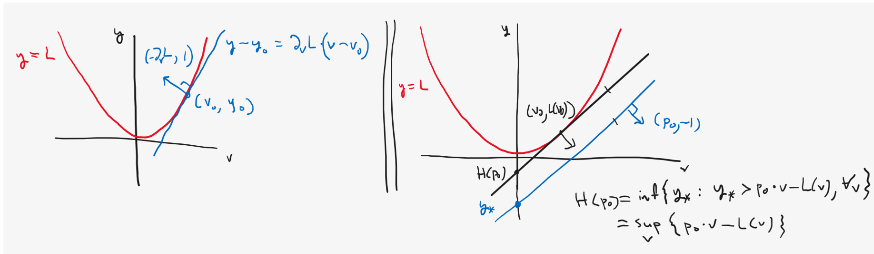


Figure 42. The graph, $v \mapsto (v, L(v)) \in \mathbb{R}^n \times \mathbb{R}$, of the Lagrangian (with *fixed* $q$ and $L(q,v) =: L(v)$) may be embedded via an affine chart into $\mathbb{P}^{n+1}$ as $v \mapsto (v : L(v) : 1)$. The set of hyperplanes in $\mathbb{P}^{n+1}$ is denoted $\mathbb{P}^{(n+1)*}$ and is itself a projective space, with $(P_1 : \ldots : P_{n+2})$ representing the hyperplane $P_1V_1 + \ldots + P_nV_n = P_{n+1}V_{n+1} + P_{n+2}V_{n+2}$. The dual hypersurface in $\mathbb{P}^{(n+1)*}$ consisting of tangent planes to the graph of $L$ is given by $(p : H(p) : 1)$ where $p = \partial_v L$, $H = \partial_v L \cdot v - L$. Hyperplanes in $\mathbb{R}^n \times \mathbb{R}$ with a *fixed* normal direction $(p_o, -1)$ may be parametrized as: $y + y_* = p_o \cdot v$. When $L$ is convex, the minimal value of $y_*$ for which $y + y_* = p_o \cdot v$ is disjoint from the graph of $L$ gives the tangent plane, $y + p_o \cdot v_o - L(v_o) = \partial_v L(v_o) \cdot v$, to $L$ at $(v_o, L(v_o))$ where $p_o = \partial_v L(v_o)$. At this point, $y_* = H(p_o)$, and we have $H(p_o) = \sup_v\{p_o \cdot v - L(v))\}$ (the Fenchel-Young inequality).

$(v, L(q,v)) \subset \mathbb{R}^n \times \mathbb{R}$ as a subset of $\mathbb{P}^{n+1}$ represented in an affine chart. Its dual is a subset of $\mathbb{P}^{(n+1)*}$, which in an affine chart is the graph of the Hamiltonian, $(p, H(q,p)),\quad p = \partial_v L,\quad H(q,p) = p \cdot v - L.$

A number of properties of the Legendre transformation follow from this description. For example: the Legendre transform of the Hamiltonian is the Lagrangian (projective duality). That is, for $v = \partial_p H$, one has $L = p \cdot v - H$. Also, one has the *Fenchel-Young inequality*:

$$H(q,p) + L(q,v) \geq p \cdot v, \quad \forall p, v \in \mathbb{R}^n$$

with equality when $p = \partial_v L$ (or $v = \partial_p H$).

With these considerations, we remark that mathematically it is natural to consider $p$ as a 'cotangent vector': an element of the dual space $\mathbb{R}^{n*}$ or linear map $p : \mathbb{R}^n \to \mathbb{R}$. Heuristically, it is also somewhat natural from a physical standpoint to think of momentum as a cotangent vector. Namely, we think of 'momentum' as a quality of a given trajectory: changing the momentum requires acting on the object with a force. Now, for an initial object at rest –which we will think of as having zero momentum– in a given configuration $q$, if we act on the object with a force at a given instant, it will cause a change in the trajectory to one passing through $q$ with some velocity $\dot{q}$. For a fixed configuration $q$, may then think of a 'momentum at $q$' as given by some function $\mathbb{R}^n \ni \dot{q} \overset{\mu}{\mapsto} \mathbb{R}$ with $\mu(0) = 0$. Linearizing, we have a cotangent vector $p : \mathbb{R}^n \to \mathbb{R}, \dot{q} \mapsto d\mu_0(\dot{q})$.

Now let us sketch some proofs of the properties we have stated above. The Hamiltonian formalism is most efficiently described in the language of manifolds and differential forms.[1] We will present our derivations and statements in this general 'coordinate free' language (it is possible to only focus on $\mathbb{R}^{2n}$ and avoid this formalism...but it is good to practice).

*proofs:* Let $M \ni q$ be the configuration space, and $TM \ni (q,v)$ the state space (tangent bundle of $M$). Consider a Lagrangian $L : TM \to \mathbb{R}$, for which $L : T_qM \to \mathbb{R}$ is convex[2]. Then the Legendre transform[3]

$$TM \overset{\mathscr{L}}{\to} T^*M, \quad (\mathscr{L}(v), u) := \frac{d}{d\varepsilon}|_{\varepsilon=0} L(q, v + \varepsilon u)$$

[1]These are an efficient notational apparatus well worth learning. See for example ch. 4, §18 and ch. 7 of Arnold's book for a quick introduction, or for example, Bachman's book *A geometric approach to differential forms*, or these notes.

[2]For mechanical systems, this is the case. We have $L(q,v) = \frac{\|v\|^2}{2} - V(q)$, where $\|\cdot\|$ is a Riemannian metric on $M$ and $V : M \to \mathbb{R}$ a function on $M$.

[3]A special case of what is called the *fiber derivative*, defined in the same way for general vector bundles. We use here the notation $(\cdot,\cdot) : V^* \times V \to \mathbb{R}$ for the natural pairing between a vector space and its dual.

is invertible, where $T^*M \ni (q,p)$ is the phase space (cotangent bundle of $M$). Define the Hamiltonian[1]

$$H : T^*M \to \mathbb{R}, \quad H(q,p) := \sup_{v \in T_qM} \{(p,v) - L(q,v)\}$$

and on the extended phase space, $T^*M \times \mathbb{R} \ni (q,p,t)$, consider the *Poincaré-Cartan* 1-form:

$$\alpha := \lambda - H\ dt$$

where $\lambda$ is the canonical 1-form[2] on $T^*M$. Now that we have formalized the relevant definitions, we will consider the optical analogues.

First, we examine the trajectories. Consider an extremal $q(t)$ of $L$ from $q(t_o) = q_o$ to $q(t_1) = q_1$ and let $\hat{q} \subset T^*M \times \mathbb{R}$ be its 'lift', $\hat{q}(t) = (\mathscr{L}(\dot{q}(t)), t)$. Then:

$$A(q) = \int_{t_o}^{t_1} L(\dot{q}(t))\ dt = \int_{t_o}^{t_1} (\mathscr{L}(\dot{q}), \dot{q}) - H(\mathscr{L}(\dot{q}))\ dt = \int_{\hat{q}} \alpha.$$

Consider a vector field $X$ on $T^*M \times \mathbb{R}$ with $X|_{T^*_{q_j}M \times \{t_j\}}$ tangent to $T^*_{q_j}M \times \{t_j\}$ and let $\varphi_\varepsilon : T^*M \times \mathbb{R} \to T^*M \times \mathbb{R}$ be the flow of $X$. Then $\pi(\varphi_\varepsilon(\hat{q}))$ is a variation $q_\varepsilon$ of $q$ and since $q$ is an extremal we have:

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_{\varphi_\varepsilon(\hat{q})} \alpha = \int_{\hat{q}} L_X \alpha = \int_{\hat{q}} i_X d\alpha.$$

Since $X$ is arbitrary away from the endpoints of $\hat{q}$, this implies that $i_R d\alpha = d\alpha(R, *) \equiv 0$ where $R$ is tangent to $\hat{q}$. Normalizing $R$ to agree with the velocity of $\hat{q}$ (ie imposing its $\partial_t$ component is 1), we take:

$$R = X_H + \partial_t \Rightarrow 0 = i_{X_H}\omega - dH \wedge dt(X_H + \partial_t, *) = i_{X_H}\omega + dH - dH(X_H) \wedge dt$$

$$\Rightarrow dH = -i_{X_H}\omega$$

which are Hamilton's equations for the trajectories on $T^*M$. Note that since $\omega$ is non-degenerate, $X_H$ is determined uniquely, and so as well the direction of $R$ is determined uniquely from the condition $i_R d\alpha \equiv 0$. Hence integral curves of $R$ projected to $M$ are extremals.

Next consider a function: $S : M \times \mathbb{R} \to \mathbb{R}$. Then for each fixed $t$, we have $S|_t : M \times \{t\} = M \to \mathbb{R}$, and a 1-form $dS|_t : M \to T^*M$. For:

$$\Gamma_S : M \times \mathbb{R} \to T^*M \times \mathbb{R}, \quad (q,t) \mapsto (dS|_t(q), t)$$

then we say $S$ satisfies the Hamilton-Jacobi equation when:

$$(\Gamma_S)^*\alpha = dS$$

ie, $H(dS|_t) = -\partial_t S$. Now we will show $R$ is tangent to the 'graph' $im(\Gamma_S)$ of $S$. First, since:

$$\Gamma_S^*(d\alpha) = d(\Gamma_S^*\alpha) = d(dS) \equiv 0$$

we have that $d\alpha$ restricted to vectors tangent to the graph of $S$ is identically zero. This implies that the tangent space to the graph of $S$ contains a direction in the kernel of $d\alpha$ [3]namely $R$, by our non-degeneracy observation above. Hence $R$ is tangent to the graph of a solution to the Hamilton-Jacobi equation, ie these graphs are 'invariant' (they contain trajectories of $R$).

This establishes the optical analogies. Now let us remark on some of the other structure we have seen.

[1] By our discussion of Legendre transform above: $H(q,p) = (p, \mathscr{L}^{-1}(p)) - L(q, \mathscr{L}^{-1}(p))$ and $L(q,v) = (\mathscr{L}(v), v) - H(q, \mathscr{L}(v))$.

[2] For $\xi \in T_pT^*M$, and $\pi : T^*M \to M, (q,p) \mapsto q$ standard projection, we take $(\lambda, \xi) := (p, \pi_*\xi)$. The two-form $\omega := d\lambda$ is the *standard symplectic structure* on $T^*M$.

[3] This is some exercise in linear algebra. Namely let $V$ be the $n+1$ dimensional tangent space to the graph at a point, with $W$ an $n$ dimensional complementary subspace, so that $V \oplus W$ spans the whole tangent space at this point. Consider $V \to W^*$ sending $v$ to $w \mapsto d\alpha(v,w)$. Since $V$ is $n+1$ dimensional and $W$ is $n$ dimensional, there must be a vector $0 \neq v_o \in V$ of its kernel. Hence $d\alpha(v_o, w) = 0$ for all $w \in W$. But now $d\alpha(v_o, v+w) = 0$ for any vector in the whole space, $V \oplus W$

For Noether's theorem, the key fact that the flow of a symplectic gradient preserves the symplectic form follows from:

$$L_{X_F}\omega = d_{i_{X_F}}\omega + i_{X_F}d\omega = -d(dF) = 0.$$

Defining the *Poisson brackets* by:

$$\{F, G\} := \omega(X_G, X_F) = dF(X_G)$$

Noether's theorem is a simple observation using that Poisson bracket is skew-symmetric: $0 = dF(X_H) = \{F, H\} = -\{H, F\} = -dH(X_F)$ so $F$ is a first integral of $X_H$, $dF(X_H) = 0$, iff the the flow of $X_F$ preserves $H$, $dH(X_F) = 0$. To pass from a symmetry vector field, $X$, to an integral in general may require some topological conditions on $M$ – namely that the closed 1-form $i_X\omega$ is exact.
Generating functions are special cases of the flows of symplectic gradients preserving $\omega$. Namely, for $S_\alpha : M \to \mathbb{R}$, a family of functions depending on parameters such that the graphs $\Gamma_{S_\alpha} = im(dS_\alpha) \subset T^*M$ foliate $T^*M$, then we consider the function $\chi : T^*M \to \mathbb{R}$ sending $p \in \Gamma_{S_\alpha}$ to $S_\alpha(\pi(p))$. The time 1-flow of $X_\chi$ is the symplectic transformation generated by the functions $S_\alpha$.
The time independent or fixed energy Hamilton-Jacobi equation may be written as

$$H \circ dS = E$$

where $S : M \to \mathbb{R}$ and $dS : M \to T^*M$. Then $(dS)^*\lambda = dS$ so that on $im(dS)$ we have $\omega \equiv 0$. As well $i_{X_H}\omega|_{H=E} \equiv 0$, so that by the same linear algebra argument with the Hamilton-Jacobi equation, we have that $X_H$ is tangent to $im(dS)$. The exact same proof works to show that $X_H$ is tangent to a graph, $im(\mu)$, of a closed ($d\mu = 0$) 1-form $\mu : M \to T^*M$ with $H \circ \mu = E$. Graphs of closed differential forms in $T^*M$ are examples of *Lagrangian submainfolds* of $T^*M$ – $n$-dimensional submanifolds for which $\omega \equiv 0$ on their tangent spaces. In even more generality, one has that if a Lagrangian submanifold is contained in an energy level of a Hamiltonian then it is invariant under the flow of $X_H$.
Finally, that the flow $\varphi_t$ of $X_H$ preserves $\omega$ is a stronger form of the *Liouville theorem*. That $\varphi_t^*\omega = \omega$ implies $\varphi_t^*\omega^n = \omega^n$, and $\omega^n$ is a volume form on $T^*M$ (so in $(q, p)$ coordinates the flow of $X_H$ preserves volume integrals, $\int_{\varphi_t\Omega} dp_1...dp_n dq_1...dq_n = \int_\Omega dp_1...dp_n dq_1...dq_n$). This fact is an essential starting point in the developement of statistical mechanics. $\square$

Exercises:

1. Let $\omega(u,v) = u \cdot Jv$ be the standard symplectic form on $\mathbb{R}^{2n}$.

   (a) If the linear transformation $M : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves $\omega$, show that $M^T J M = J$.

   (b) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and $Q = Aq$. Show that the 'symplectic lift', $P = (A^T)^{-1}p$ yields an $\omega$ preserving transformation –a symplectic transformation– of $\mathbb{R}^{2n} \ni (q,p)$.

2. Let $\omega$ be a skew-symmetric bilinear form on an odd dimensional vector space $V$. Show that there exists some $0 \neq v_o \in V$ such that $\omega(v_o, v) = 0$ for all $v \in V$.

3. Let $\omega(u,v) = u \cdot Jv$ be the standard symplectic form on $\mathbb{R}^{2n}$. Suppose that the vector subspace $V \subset \mathbb{R}^{2n}$ is such that $\omega(u,v) = 0$ for all $u, v \in V$. Show that $V$ has dimension at most $n$.

4. Let $\omega(u,v) = u \cdot Jv$ be the standard symplectic form on $\mathbb{R}^{2n}$. Suppose that the vector subspace $V \subset \mathbb{R}^{2n}$ has dimension $k$. Define its $\omega$-orthogonal complement as $V' = \{u \in \mathbb{R}^{2n} : \omega(u,v) = 0, \forall v \in V\}$. Show that $V'$ has dimension $2n - k$.

5. For $a > 1$ show that the Legendre transform of $L(v) = \frac{v^a}{a}$ (for $v \in \mathbb{R}, v > 0$) is $H(p) = \frac{p^b}{b}$ with $b > 1$ satisfying:
$$\frac{1}{a} + \frac{1}{b} = 1.$$
   Deduce that $pv \leq \frac{v^a}{a} + \frac{p^b}{b}$, with equality iff $v^a = p^b$.

**§13 integrable systems**

Given a Hamiltonian system, $H(q,p)$, $(q,p) \in \mathbb{R}^{2n}$, the Hamilton-Jacobi equation has led us to a scheme to 'solve' the system by seeking a change of coordinates to a Hamiltonian system, $h(P)$, $(Q,P) \in \mathbb{R}^{2n}$, having $n$ first integrals (the 'parameters' $P_1, ..., P_n$). In general –by Noether's theorem– the presence of a first integral gives the possibility to reduce the dimensions under consideration by *two*: through the process of restriction to a level set of the first integral and then by passing to a quotient under the corresponding symmetry action. Thus one expects the presence of $n$ first integrals to lead –via repeated applications of this restriction and quotient process– to a reduced description of the motion in a one-dimensional problem, which one considers to be 'solved'.

We will define a class of Hamiltonian systems –said to be *Liouville integrable*– which admit solutions in this way, namely through $n$ first integrals which 'commute':

**Liouville-Milneur-Arnold theorem:** Consider a Hamiltonian system $H(q,p)$, $(q,p) \in \mathbb{R}^{2n}$ admitting $n$ independent first integrals $F_1, ..., F_n$ in involution:[1]

$$\{F_i, F_j\} = 0.$$

If the level sets $\cap\{F_j = cst.\}$ are compact, there exist (local) symplectic coordinates[2] $(I, \theta) \in \mathbb{R}^n \times T^n$ s.t.

$$H = h(I)$$

is a function only of $I \in \mathbb{R}^{2n}$.

*proof:* Let $X_1, ..., X_n$ be the symplectic gradients of $F_1, ..., F_n$ with flows $\varphi_t^n$. Then:

$$[X_i, X_j] = X_{\{F_j, F_i\}} = 0$$

where $[X_i, X_j]$ is the *Lie bracket*[3] of $X_i, X_j$. Hence the flows commute: $\varphi_t^i \circ \varphi_s^j = \varphi_s^j \circ \varphi_t^i$. Now, by assumption, a level set $\Sigma := \cap\{F_j = cst.\}$ is a compact $n$-dimensional submanifold of $\mathbb{R}^{2n}$. Since $\{F_i, F_j\} = 0$, the symplectic gradients, $X_j$, are tangent to $\Sigma$ and we have an $\mathbb{R}^n$ action on $\Sigma$ by:

$$\vec{t} \cdot x := \varphi_{t_1}^1 \circ ... \circ \varphi_{t_n}^n(x), \quad \vec{t} = (t_1, ..., t_n) \in \mathbb{R}^n, \quad x \in \Sigma.$$

Since $\Sigma$ is compact, there exists $\varepsilon > 0$ s.t. $\vec{t} \cdot x \neq x$ for all $0 < |\vec{t}| < \varepsilon$. It follows that upon fixing a 'base-point' $x_o \in \Sigma$ we have an onto map:

$$\mathbb{R}^n \to \Sigma, \quad \vec{t} \mapsto \vec{t} \cdot x_o.$$

This map is not 1-1 since $\Sigma$ is compact, while $\mathbb{R}^n$ is not compact. Let:

$$\Gamma := \{\vec{t} \in \mathbb{R}^n : \vec{t} \cdot x_o = x_o\}$$

then $\Gamma$ is a discrete subgroup of $\mathbb{R}^n$, with $\Gamma = span_{\mathbb{Z}}\{v_1, ..., v_n\}$ for some basis $v_j$ of $\mathbb{R}^n$. Now:

$$T^n = \mathbb{R}^n / \mathbb{Z}^n \ni (\theta_1, ..., \theta_n) \mapsto (\theta_1 v_1 + ... + \theta_n v_n) \cdot x_o \in \Sigma$$

is a bijective map identifying $\Sigma$ with the $n$-torus $T^n$. By doing this for each level set, we obtain local coordinates on $\mathbb{R}^{2n}$, by $(q,p) \mapsto (F(q,p), \theta(q,p)) \in \mathbb{R}^n \times T^n$. In general these coordinates are not symplectic, however they may be made symplectic by an appropriate 'change of basis'.

[1] Here we use the notation of Poisson brackets: $\{F, G\} = dF(X_G) = \omega(X_G, X_F)$. Of course $H$ itself is always a first integral and may be taken for $F_1 = H$. By *independent* first integrals we mean that at each point, $dF_1, ..., dF_n$ are independent covectors (ie the gradients $\nabla F_1, .., \nabla F_n$ are independent vectors).

[2] Called *action-angle* coordinates. Here $T^n$ is the $n$-torus, and that $(I, \theta)$ are symplectic means $\omega = dp \wedge dq = dI \wedge d\theta$.

[3] It may be defined by $[X, Y] = L_X Y = \frac{d}{dt} \varphi_*^{-t} Y$ where $\varphi_t$ is the flow of $X$. The relation with Poisson brackets follows from the identities: $d\{F, G\} = L_{X_G} dF = -L_{X_G}(i_{X_F}\omega) = -i_{[X_G, X_F]}\omega$ (using that $L_{X_G}\omega = 0$). The identity, $X_{\{F,G\}} = [X_G, X_F]$ implies the *Jacobi identity* for Poisson brackets: $\{\{X_F, X_G\}, X_H\} + \{\{X_G, X_H\}, X_F\} + \{\{X_H, X_F\}, X_G\} = 0$.

Let $v_j = \sum_k a_j^k e_k$, and[1] $I_j := \oint_{\gamma_j} \lambda$ where $\gamma_j$ are the 'cycles' on the tori $\Sigma$. Then $dI_j(\partial_{\theta_k}) = 0$, and

$$dI_j(\partial_{F_k}) = \oint_{\gamma_j} L_{\partial_{F_k}} \lambda = \oint_{\gamma_j} i_{\partial_{F_k}} \omega = \int_0^1 \omega(\partial_{F_k}, \partial_{\theta_j})\, d\theta_j = \sum_\ell a_j^\ell \int_0^1 \omega(\partial_{F_k}, X_\ell)\, d\theta_j$$

$$= \sum_\ell a_j^\ell \int_0^1 dF_\ell(\partial_{F_k})\, d\theta_j = a_j^k$$

$$\Rightarrow dI_j = \sum_k a_j^k dF_k \Rightarrow -i_{\partial_{\theta_j}} \omega = dI_j$$

so the symplectic gradient, $-J\nabla I_j$, of $I_j$ is $\partial_{\theta_j}$. Hence $\omega(\partial_{I_j}, \partial_{I_k}) \sim \nabla I_j \cdot J\nabla I_k = \nabla I_k \cdot \partial_{\theta_j} = dI_k(\partial_{\theta_j}) = 0$. Since as well $\omega(\partial_{I_j}, \partial_{\theta_k}) = \delta_{jk}, \omega(\partial_{\theta_j}, \partial_{\theta_k}) = 0$, we have in total that indeed $\omega = dI \wedge d\theta$. $\square$

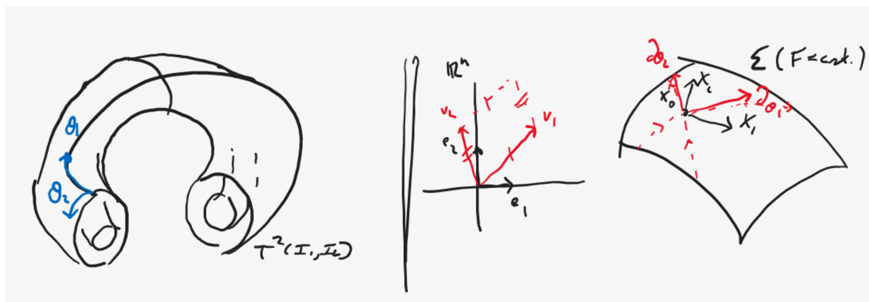


Figure 43. A Liouville integrable problem with compact level sets has the geometric structure of a foliation into tori. The flows under the symplectic gradients $X_j$ generate an $\mathbb{R}^n$ action on the level sets, and determine a lattice $\Gamma \subset \mathbb{R}^n$ by which the level set is the quotient, $\mathbb{R}^n/\Gamma \cong T^n$.

The systems we have met upto this point have been Liouville integrable problems:

- Systems with one degree of freedom are Liouville integrable with $F_1 = H$. Compact level sets of a function on the plane are circles.
- Central force problems have the energy and momentum integrals $F_1 = H, F_2 = C$ in involution. As we saw in figure 14, the compact energy-momentum levels are tori.
- Geodesics on surfaces of revolution or the spherical pendulum admit the energy and 'momentum about the axis of revolution' as integrals in involution. As with the central force problems, one can check that the compact level sets are topologically tori.
- The (free) rigid body admits the energy, $F_1 = H$, and components $C_1, C_2, C_3$ of its angular momentum vector as integrals. Here we have $\{H, C_j\} = 0$, but the components $C_j$ are not in involution, in fact one finds $\{C_1, C_2\} = C_3, \{C_2, C_3\} = C_1, \{C_3, C_1\} = C_2$. The norm squared $C^2 = C_1^2 + C_2^2 + C_3^2$ of the angular momentum is in involution and one may take $F_2 = C_1, F_3 = C^2$ for three integrals in involution.
- For the Lagrange top, the energy and two additional momentum integrals are three integrals in involution. That is, the Hamiltonian is: $H = \frac{p_\theta^2}{2I} + \frac{(p_\varphi - \cos\theta p_\psi)^2}{2I\sin^2\theta} + \frac{p_\psi^2}{2I_3} - gML\cos\theta$, with $\omega = dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi + dp_\psi \wedge d\psi$, and the commuting first integrals $H, p_\varphi, p_\psi$.

[1] These $I_j$'s are well-defined since $X_j$ span the tangent space to $\Sigma$ and $0 = \omega(X_i, X_j) = d\lambda(X_i, X_j)$. Moreover, as functions $I_j : \mathbb{R}^{2n} \to \mathbb{R}$ their values depend only on the values of the first integrals $F_j$, ie $I_j(F_1, ..., F_n)$. By the independence assumption, we have inversely that $F_j(I_1, ..., I_n)$. In particular $F_1 = H = h(I)$.

- The symplectic coordinates $(Q,P)$ generated by a 'general solution' $H(q,\partial_q S(q,P)) = h(P)$ of the Hamilton-Jacobi equation depending on $n$ parameters yields $n$ integrals $P_1,...,P_n$ in involution.
- Seperation of variables may be applied in certain coordinates to 'integrate' the 2-center problem and the problem of geodesics on a quadratic surface by the Hamilton-Jacobi method.

  The (Newtonian) 2-center problem considers the motion of a point mass $q \in \mathbb{R}^2$ subject to inverse square attraction by two *fixed* points, 'centers'. In Hamiltonian form:

$$H = \frac{|p|^2}{2} - \frac{m_1}{r_1} - \frac{m_2}{r_2}, \quad \omega = dp \wedge dq,$$

  where $r_j$ are the distances from $q$ to the centers. Wlog, consider when the distance between the centers is 2 and they are located at the points with Cartesian coordinates $(\pm 1, 0)$. The seperation of the Hamilton-Jacobi equation will be found using elliptic coordinates.

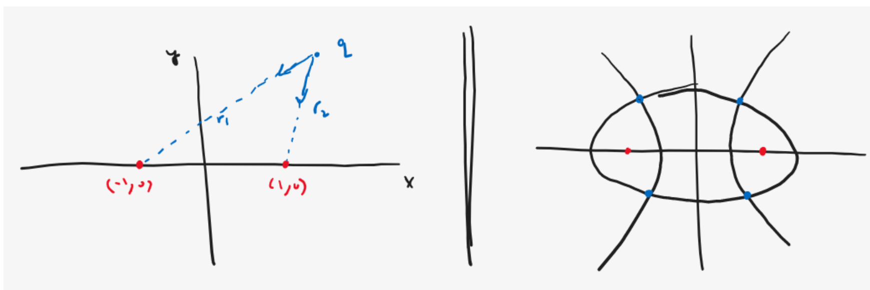


Figure 44. The two-center problem considers a point mass in the plane subject to attraction by two fixed centers. One may solve the Hamilton-Jacobi equation in elliptic coordinates: an ellipse and hyperbola with the same foci located at the fixed centers intersect in 4-points. Parametrizations of such conics may be used to parametrize quadrants of the plane.

Namely, for each fixed $\mu \in (1,\infty)$, the curve $\frac{x^2}{\mu} + \frac{y^2}{\mu-1} = 1$ is an ellipse with foci at the centers having semi-major axis $a^2 = \mu$. Likewise, for each fixed $\nu \in (0,1)$, the curve $\frac{x^2}{\nu} - \frac{y^2}{1-\nu} = 1$ is a hyperbola with foci at the centers having semi-major axis $a^2 = \nu$. Quadrants in the $(x,y)$ plane are coordinatized by sending $(\mu,\nu) \in (1,\infty) \times (0,1)$ to the intersection point of the conics with parameters $\mu,\nu$. Explicitely:

$$x^2 = \mu\nu, \quad y^2 = (\mu-1)(1-\nu).$$

In fact, the coordinates may be made 1:1 by taking $\mu = \xi^2, \nu = \eta^2$ with the signs of $\xi, \eta$ yielding the corresponding quadrants. One has then that $\eta = r_1 - r_2, \quad |\xi| = r_1 + r_2$, so that the potential is:

$$-\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) = -\frac{2(m_1+m_2)|\xi| + 2(m_2-m_1)\eta}{\xi^2 - \eta^2}.$$

For the kinetic term, consider a curve $q(\xi(t),\eta(t)) \in \mathbb{R}^2$. Then:

$$\dot{q} = q_\xi \dot{\xi} + q_\eta \dot{\eta}$$

where, $q_\xi = (x_\xi, y_\xi) = (\frac{\xi\eta^2}{x}, \frac{\xi(1-\eta^2)}{y}), \quad q_\eta = (\frac{\xi^2\eta}{x}, \frac{\eta(1-\xi^2)}{y})$, and so the Kinetic energy is:

$$\frac{1}{2}|\dot{q}|^2 = \frac{\xi^2 - \eta^2}{2}\left(\frac{\dot{\xi}^2}{\xi^2 - 1} + \frac{\dot{\eta}^2}{1-\eta^2}\right)$$

With momenta, $p_\xi = \frac{\xi^2-\eta^2}{\xi^2-1}\dot{\xi}, \quad p_\eta = \frac{\xi^2-\eta^2}{1-\eta^2}\dot{\eta}$ and finally:

$$H = \frac{1}{2(\xi^2-\eta^2)}\left((\xi^2-1)p_\xi^2 + (1-\eta^2)p_\eta^2\right) - \frac{k_1|\xi| + k_2\eta}{2(\xi^2-\eta^2)}, \quad \omega = dp_\xi \wedge d\xi + dp_\eta \wedge d\eta$$

with $k_1 = 4(m_1 + m_2), k_2 = 4(m_2 - m_1)$. Now, our Hamilton-Jacobi equation for $S(\xi, \eta)$ is:

$$cst. = E = \frac{1}{2(\xi^2 - \eta^2)} \left((\xi^2 - 1)S_\xi^2 + (1 - \eta^2)S_\eta^2\right) - \frac{k_1|\xi| + k_2\eta}{2(\xi^2 - \eta^2)}$$

$$\Rightarrow (\xi^2 - 1)S_\xi^2 - k_1|\xi| - 2E\xi^2 = (\eta^2 - 1)S_\eta^2 + k_2\eta - 2E\eta^2$$

Seeking solutions by seperation of variables, in the form $S(\xi, \eta) = f(\xi) + g(\eta)$, gives the system of first order ode's:

$$(\xi^2 - 1)(f'(\xi))^2 - k_1|\xi| - 2E\xi^2 = c_1, \quad (\eta^2 - 1)(g'(\eta))^2 + k_2\eta - 2E\eta^2 = c_1$$

for $c_1$ a constant. Setting $c_2 := 2E$ as another constant, we have:

$$f(\xi; c_1, c_2) = \int \sqrt{\frac{c_1 + c_2\xi^2 + k_1|\xi|}{\xi^2 - 1}}\, d\xi, \quad g(\eta; c_1, c_2) = \int \sqrt{\frac{c_1 + c_2\eta^2 - k_2\eta}{\eta^2 - 1}}\, d\eta$$

defining a general solution to the Hamilton-Jacobi equation $S(\xi, \eta; c_1, c_2) = f(\xi; c_1, c_2) + g(\eta; c_1, c_2)$ depending on two parameters $c_1, c_2$:

$$H(\xi, \eta, \partial_\xi S, \partial_\eta S) = \frac{c_2}{2}.$$

Thus one has the energy first integral $c_2 = \frac{H}{2}$, and the new first integral:

$$c_1 = (\eta^2 - 1)p_\eta^2 + k_2\eta - 2H\eta^2 = (\xi^2 - 1)p_\xi^2 - k_1|\xi| - 2H\xi^2.$$

Moreover, the solutions with given first integral values are defined implicitly as functions of time through:

$$t + t_o = 2\left(\partial_{c_2} f(\xi(t); c_1, c_2) + \partial_{c_2} g(\eta(t); c_1, c_2)\right)$$

$$cst. = \partial_{c_1} f(\xi(t); c_1, c_2) + \partial_{c_1} g(\eta(t); c_1, c_2).$$

Geodesics on a quadratic surface are handled similarly with 'ellipsoidal coordinates'. Let

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$$

for fixed non-zero $a > b > c$ (with $a > 0$) define a quadratic surface in $\mathbb{R}^3$. One parametrizes 'octants' in $\mathbb{R}^3$ by sending $(u, v, w) \in (-\infty, c) \times (c, b) \times (b, a)$ to the intersection points of the ellipsoid, 1-sheeted hyperboloid and 2-sheeted hyperboloid given by the equations:

$$\frac{x^2}{a-u} + \frac{y^2}{b-u} + \frac{z^2}{c-u} = 1, \quad \frac{x^2}{a-v} + \frac{y^2}{b-v} + \frac{z^2}{c-v} = 1, \quad \frac{x^2}{a-w} + \frac{y^2}{b-w} + \frac{z^2}{c-w} = 1$$

respectively. Explicitely, one has:

$$x^2 = \frac{(a-u)(a-v)(a-w)}{(a-b)(a-c)}, \quad y^2 = \frac{(b-u)(b-v)(b-w)}{(b-a)(b-c)}, \quad z^2 = \frac{(c-u)(c-v)(c-w)}{(c-b)(c-a)}.$$

The norm squared of the velocity of a curve in these coordinates is given by:

$$\frac{1}{4}\left(\frac{(v-u)(w-u)}{(a-u)(b-u)(c-u)}\dot{u}^2 + \frac{(u-v)(w-v)}{(a-v)(b-v)(c-v)}\dot{v}^2 + \frac{(u-w)(v-w)}{(a-w)(b-w)(c-w)}\dot{w}^2\right).$$

Now, consider the case when the original surface is an ellipsoid, ie $c > 0$, so that it is parametrized by $v, w$ with $u = 0$. Then the norm squared of velocity (twice kinetic energy) on the ellipsoid is:

$$\frac{(w-v)}{4}\left(\frac{v}{(a-v)(b-v)(v-c)}\dot{v}^2 + \frac{w}{(a-w)(w-b)(w-c)}\dot{w}^2\right)$$

given by the Hamiltonian system:

$$H = \frac{1}{(w-v)} \left( \frac{(a-v)(b-v)(v-c)}{v}\ p_v^2 + \frac{(a-w)(w-b)(w-c)}{w}\ p_w^2 \right)$$

with $\omega = dp_v \wedge dv + dp_w \wedge dw$. A general solution to the Hamilton-Jacobi equation $c_1 = H(v, w, S_v, S_w)$ may be found by seperation of variables with $S(v, w; c_1, c_2) = f(v; c_1, c_2) + g(w; c_1, c_2)$, and:[1]

$$f(v; c_1, c_2) = \int \sqrt{\frac{(c_2 - c_1 v)v}{(a-v)(b-v)(v-c)}}\ dv, \quad g(w; c_1, c_2) = \int \sqrt{\frac{(c_1 w - c_2)w}{(a-w)(w-b)(w-c)}}\ dw.$$

As before, $c_1 = H$ is the energy integral, while we have the new integral of motion:

$$c_2 = \frac{(a-v)(b-v)(v-c)}{v} p_v^2 + Hv = Hw - \frac{(a-w)(w-b)(w-c)}{w} p_w^2.$$

This integral – like the Clairaut integral for surfaces of revolution – may be described geometrically and used to sketch the geodesics. Namely we have:

$$cst. = C = \frac{c_2}{H} = w\cos^2\theta + v\sin^2\theta = w\sin^2\varphi + v\cos^2\varphi$$

where $\theta$ is the angle between the tangent to the geodesic and the lines $w = cst.$ (intersections of ellipsoid with 2-sheeted hyperboloids) and $\varphi$ the angle between the tangent to the geodesic and the lines $v = cst.$ (intersections of the ellipsoid with 1-sheeted hyperboloids).

We will explain in §15 a geometric mechanism that 'blocks' Liouville integrability, and present some examples of systems that are *not* Liouville integrable, namely certain forced pendulums and the restricted three-body problem.

[1] These integrals may be given explicitely in terms of elliptic functions.

**§14 symplectic reduction**[1]

The 'restrict and quotient' process we have alluded to for reducing systems in the presence of symmetry (and corresponding first integral) may be formulated in the language of the Hamiltonian formalism via the process of symplectic reduction.

Consider a Hamiltonian system, $H(q,p), \omega = dp \wedge dq$. The symmetries of the system are transformations $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserving $H$ and $\omega$. Under composition the set of all symmetries form a group. In general this group may consist only of the identity element or be a discrete collection of transformations. However we have seen in Noether's theorem that 1-parameter families, $\varphi_t$ with $\varphi_o = id$, of symmetries are related to first integrals. The systems to which one applies a symplectic reduction admit 1-parameter families of symmetries, so that the group of symmetries of the system is a *Lie group*[2]. We will call its connected component of the identity $G$.

Now, as opposed to Noether's theorem where we consider a single 1-parameter family, we will now try to consider the integrals associated to all possible 1-parameter families. Consider a curve $g(t) \in G$ with $g(0) = id$ (some 1-parameter family of symmetries). We take

$$\xi = \frac{d}{dt}|_{t=0} g(t) \in \mathfrak{g} := T_{id}G.$$

There is a corresponding vector field $X_\xi$ on $\mathbb{R}^{2n}$ defined by

$$X_\xi(x) = \frac{d}{dt}|_{t=0} g(t)(x).$$

Now, by assumption, the vector field $X_\xi$ has a flow which preserves $\omega$, ie $0 = L_{X_\xi}\omega = d(i_{X_\xi}\omega)$. Since we are working on $\mathbb{R}^{2n}$, every closed 1-form is exact, so there exists a corresponding first integral, $J_\xi : \mathbb{R}^{2n} \to \mathbb{R}$, of which $X_\xi$ is its symplectic gradient:

$$dJ_\xi = -i_{X_\xi}\omega.$$

Thus, we have obtained a map, $\xi \mapsto J_\xi$, from $\mathfrak{g}$ to first integrals of the system. Note however that these first integrals $J_\xi$ are defined upto addition of constants. One would thus like a more explicit definition of the $J_\xi$'s in a way that makes clear they have been chosen in a consistent way. Note that $i_{X_\xi}\omega = i_{X_\xi}d\lambda = L_{X_\xi}\lambda - d(i_{X_\xi}\lambda)$, where $d\lambda = \omega$. Thus, one would like to take $J_\xi = i_{X_\xi}\lambda$, which works provided the symmetries as well preserve $\lambda$ (ie $L_{X_\xi}\lambda = 0$). We will restrict our attention to these types of 'exact' symmetries of the system to avoid possible ambiguities in first integrals defined upto constants.

This having been done –with $J_0 = 0$– one may express fixing the values of these first integrals as fixing an element $\mu \in \mathfrak{g}^*$ of the dual space to $\mathfrak{g}$, that is $\mu : \mathfrak{g} \to \mathbb{R}$. Namely fixing the values $J_\xi = cst.$, is equivalent to fixing the values of $\mu(\xi) := J_\xi(q,p)$, so that all the first integrals may be expressed in a *moment map*:

$$J : \mathbb{R}^{2n} \to \mathfrak{g}^*, \quad (J(q,p), \xi) = J_\xi(q,p).$$

Fixing the value of $J$ fixes the values of the first integrals of the system.

**Symplectic reduction (Meyer-Marsden-Weinstein reduction):** Let $G$ be the identity component of symmetries of the Hamiltonian system $H, \omega = d\lambda$ preserving $\lambda$. Then there exists a moment map:

$$J : \mathbb{R}^{2n} \to \mathfrak{g}^*$$

s.t. for each fixed $\mu \in \mathfrak{g}^*$, the symplectic gradient, $X_H$, of $H$ is tangent to the level set $J^{-1}(\mu)$. Moreover, for $G_\mu$, the elements of $G$ preserving $J^{-1}(\mu)$, the quotient space –provided it is a manifold–

$$J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu =: P_\mu$$

has a symplectic structure, $\omega_\mu$, and Hamiltonian $H_\mu$ s.t. the projected trajectories of $X_H$ from $J^{-1}(\mu)$ are sent to the trajectories of the reduced Hamiltonian system, $X_{H_\mu}$.

[1] The relevant section of Arnold is appendix 5.
[2] For example the matrix groups, $\mathrm{SO}_3, \mathrm{SL}_2(\mathbb{R})$ are Lie groups.

*proof:* As outlined above, for $\xi \in \mathfrak{g}$, let $X_\xi$ be the vector field[1] on $\mathbb{R}^{2n}$ which is the symplectic gradient of the function $J_\xi = i_{X_\xi}\lambda$. Define $J : \mathbb{R}^{2n} \to \mathfrak{g}^*$ by $(J(x), \xi) = J_\xi(x)$. By assumption, the flow of $X_\xi$ preserves $H$, ie $0 = dH(X_\xi) = \{H, J_\xi\} = -\{J_\xi, H\} = -dJ_\xi(X_H)$. So, for $\varphi_t$ the flow of $X_H$, we have $J_\xi(\varphi_t(x)) = J_\xi(x)$, ie the functions $J_\xi : \mathbb{R}^{2n} \to \mathbb{R}$ are first integrals. Now, fix $\mu \in \mathfrak{g}^*$ and let $\xi \in \mathfrak{g}, x \in J^{-1}(\mu)$. Then $(J(\varphi_t(x)), \xi) = J_\xi(\varphi_t(x)) = J_\xi(x) = (\mu, \xi)$. Since $\xi$ was arbitrary, we have $J(\varphi_t(x)) = J(x) = \mu$, ie the level set $J^{-1}(\mu)$ are invariant.
Now, let $G_\mu$ be the elements of $G$ preserving $J^{-1}(\mu)$ and set[2] $P_\mu := J^{-1}(\mu)/G_\mu$ for the quotient space (the space of orbits of $G_\mu$ on $J^{-1}(\mu)$) with projection $\pi : J^{-1}(\mu) \to P_\mu$. Let us first consider the symplectic structure $\omega_\mu$ on $P_\mu$. Observe that the tangent spaces to the level set $J^{-1}(\mu)$ and the tangent spaces to the orbits $G \cdot x$ are $\omega$-orthogonal complements, since, for $X \in T_x J^{-1}(\mu)$ and $X_\xi(x)$ tangent to the group orbit $G \cdot x$, we have:

$$\omega(X, X_\xi) = dJ_\xi(X) = 0$$

since $J_\xi$ is constant along $X \in T_x J^{-1}(\mu)$. It follows that the tangent spaces to the orbits $G_\mu \cdot x$, as the intersections of the orbit $G \cdot x$ with $J^{-1}(\mu)$, satisfy $\omega(X, X_\xi) = \omega(X_\eta, X_\xi) = 0$ for any $X$ tangent to $J^{-1}(\mu)$ and $X_\xi, X_\eta$ tangent to $G_\mu \cdot x$. Hence, for $x \in J^{-1}(\mu)$ and $X, Y \in T_x J^{-1}(\mu)$ we have a well-defined:

$$\omega_\mu(\bar{X}, \bar{Y}) := \omega(X, Y)$$

where $\bar{X}, \bar{Y}$ are the projections of $X, Y$. This $\omega_\mu$ determines a symplectic structure on $P_\mu$. It is non-degenerate since if $\omega_\mu(\bar{X}, *) = 0$, then $\omega(X, Y) = 0$ for every $Y$ tangent to the level set, hence $X$ is tangent to an orbit and its projection $\bar{X}$ is zero. It is closed since $\pi^*\omega_\mu = \omega$ is closed and $\pi_*$ is onto.
The reduced Hamiltonian is given by $H_\mu(\pi(x)) = H(x)$ where $x \in J^{-1}(\mu)$ and is well-defined since $H$ is invariant along the orbits $G_\mu$. Finally, we have:

$$dH_\mu(\bar{X}) = dH(X) = \omega(X, X_H) = \omega_\mu(\bar{X}, \pi_* X_H)$$

for any $\bar{X} = \pi_* X$, so that $X_{H_\mu} = \pi_* X_H$, ie an integral curve of $X_{H_\mu}$ is the projection of an integral curve of $X_H$ lying in $J^{-1}(\mu)$. $\square$

This general reduction process permits one to 'count dimensions' in a precise way and gauge how reasonable a problem will be given the symmetries one is aware of. It may be presented concisely in a commutative diagram:

$$\begin{tikzcd} & \mathbb{R}^{2n} \\ J^{-1}(\mu) \arrow[ur, "\iota"] \arrow[dr, "\pi"'] & \\ & P_\mu \end{tikzcd}$$

With $\iota$ the inclusion and $\pi$ the projection to the quotient. Then:

$$i^*\omega = \pi^*\omega_\mu, \quad i^*H = \pi^*H_\mu, \quad \pi_* X_H = X_{H_\mu}.$$

Examples:

- Any transformation of $\mathbb{R}^n$, $q \mapsto \varphi(q)$ lifts to a $\lambda = p \cdot dq$ preserving symplectic tranformation of $\mathbb{R}^{2n} \ni (q, p)$ by $\hat{\varphi}(q, p) = (\varphi(q), (d_q\varphi^{-1})^t p)$.

[1]That the group $G$ acts on $\mathbb{R}^{2n}$ is expressed through a map $a : G \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, (g, x) \mapsto g \cdot x$. Then for $i_x : G \to G \times \mathbb{R}^{2n}, g \mapsto (g, x)$ we have $X_\xi(x) = d_{id}(a \circ i_x)(\xi)$. Note that by linearity of differentials, we have: $X_{\xi + c\eta} = X_\xi + cX_\eta$.

[2]In general one may need to add conditions for $P_\mu$ to be a manifold. For example, when $\mu$ is a regular value of $J$ and $G_\mu$ is compact acting freely on $J^{-1}(\mu)$.

- The Lagrange top admits symmetries by independent rotations about symmetry axes, $G = S^1 \times S^1$. The actions commute, so that on the fixed integral level set (4-dimensional) one has a quotient by the full 2-dimensional symmetry group taking one to a Hamiltonian system on a 2-dimensional space, 'solvable' by taking level sets of the reduced Hamiltonian.

- For the free rigid body, we begin in the 6-dimensional $T^*\mathrm{SO}_3$ with three dimensional symmetry group $\mathrm{SO}_3$ and first integrals the components of the angular momentum vector. Fixing the angular momentum we have a 3-dimensional invariant set on which acts the 1-dimensional rotation group of rotations about this fixed angular momentum axis taking us to a 2-dimensional quotient space on which the solutions may be described by taking level sets of the reduced Hamiltonian.

- For the problem of $n$-point masses in space under Newtonian attraction –the *n-body problem*– one begins with a $6n$-dimensional problem. Fixing the center of mass at the origin one has a $6(n-1)$ dimensional problem with rotational $\mathrm{SO}_3$ symmetry and angular momentum vector conserved. Fixing the angular momentum vector reduces the dimension by 3 (the three components of the vector) and one may quotient by the rotations having this momentum vector as axis to arrive at a $6(n-1)-4$ dimensional problem. When $n = 3$ and one considers the bodies to be located in a fixed plane, one finds a 6-dimensional problem.

- There is much more structure to symplectic reduction and momentum maps than we have presented here. For instance –with exact ($\lambda$ preserving) symmetries– there is a compatability (equivariance) between the $G$-action on $\mathbb{R}^{2n}$ and the co-adjoint action:

$$J(g \cdot x) = Ad^*_{g^{-1}} J(x).$$

  One may identify $P_\mu = J^{-1}(\mu)/G_\mu \cong J^{-1}(\mathcal{O}_\mu)/G$ where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the co-adjoint orbit of $\mu$. See for example appendices 2, 5 of Arnold.

## §15 perturbation theory

Chronologically, the symplectic form and Hamilton's equations appeared before Hamilton's optical analogy in a somewhat more disguised form in works on perturbation theory –in particular the works of Lagrange and Poisson[1]. This appearance came via the method of *variation of constants*. Consider an ode:

$$\dot{x} = v_o(x)$$

to which a general solution $x(t) = \varphi(t; x_o)$, with initial condition $x(0) = \varphi(0; x_o) = x_o$ is known. Then one may approach an ode of the form

$$\dot{x} = v_o(x) + v_1(x)$$

By seeking solutions of the form $x(t) = \varphi(t; x_o(t))$, in which the previous constants of integration, $x_o$, are now allowed to depend on time. Solving the new ode is reduced to an ode for $x_o(t)$:

$$(\partial_{x_o}\varphi)\dot{x}_o = v_1(\varphi(t; x_o)).$$

Provided a solution to this auxillary ode for $x_o(t)$ may be found, one thus obtains a solution to the 'perturbed' ode $\dot{x} = v_o(x) + v_1(x)$. When $v_1$ is 'small' then $x_o(t)$ changes slowly and this solution $x(t) = \varphi(t; x_o(t))$ to the perturbed system may be visualized as a slowly deforming family of solutions to the original ode $\dot{x} = v_o(x)$, the deformations being characterized through the changes in initial conditions $x_o(t)$.

The method of variation of constants was applied by Lagrange to describe more accurately planetary orbits in the solar system. The dominant effect on a planet's orbit is the influence of the massive sun, whose solution is known explicitly –the solution of the Kepler problem, $\ddot{q} = -GM\frac{q}{|q|^3}$. As constants of integration for the problem one may take parameters determining an elliptic (planetary motions) orbit and an initial position on said elliptic orbit. These constants are called the *orbital elements*, for the planar problem written:

$$a, e, g, T.$$

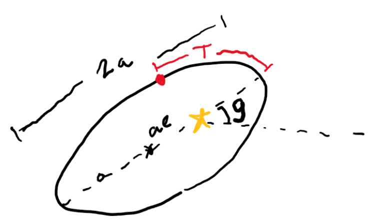


Figure 45. The orbital elements are coordinates on position and velocity space. To a given position and velocity one assigns the pointed Kepler orbit that satisfies those initial conditions. The parameter $a$ is the semi-major axis, $e$ the eccentricity, $g$ the *argument of pericenter* (closest approach) and $T$ the *time to pericenter* –giving the current position of the planet along the orbit.

To take into account the effects of the gravitational influence of other bodies –eg the small perturbations due to the forces of mutual interactions between planets– Lagrange considered these perturbing forces as derived from a potential function, $\Omega(a, e, g, T)$, and wrote the equations for the variations of the constants in the form:

$$\partial_a\Omega = (e,a)\ \dot{e} + (g,a)\ \dot{g} + (T,a)\ \dot{T}, \quad \partial_e\Omega = (a,e)\ \dot{a} + (g,e)\ \dot{g} + (T,e)\ \dot{T},$$
$$\partial_g\Omega = (a,g)\ \dot{a} + (e,g)\ \dot{e} + (T,g)\ \dot{T}, \quad \partial_T\Omega = (a,T)\ \dot{a} + (e,T)\ \dot{e} + (g,T)\ \dot{g}.$$

The coefficients, $(a,e), (a,g), \ldots$ etc. are called the *Lagrange parentheses*, and a remarkable symmetry in these coefficients was noticed, eg $(a,e) = -(e,a), \ldots$ etc. This anti-symmetry is due to the fact that the Lagrange

[1] J.L. Lagrange, *Memoire sur la theorie des variations des elements des planetes, et en particulier des variations des grands axes de leurs orbites.* Paris (1808). S.D. Poisson, *Sur la variation des constantes arbitraires dans les questions de mécanique.* l'école polytecnique (1809). See as well, C.M. Marle, *The inception of symplectic geometry: the works of Lagrange and Poisson during the years 1808–1810.* Letters in Mathematical Physics 90, 1-33, (2009). available here.

parentheses are the coefficients of the standard symplectic form written in a general system of coordinates, so the equations above for variation of the constants are –in disguised form– Hamilton's equations in a different system of coordinates!

We consider now general changes of coordinates. That is, for $x = (q, p) \in \mathbb{R}^{2n}$ with the standard symplectic form $\omega = dp \wedge dq$ we consider $\mathbb{R}^{2n} \xrightarrow{\varphi} \mathbb{R}^{2n}, y \mapsto \varphi(y) = x$, where $y = (y_1, ..., y_{2n})$ are a new system of coordinates. We set $A := d_y\varphi$ for the Jacobian matrix of this coordinate change.

**Lagrange parentheses:** In coordinates $\varphi(y) = (q, p)$, the symplectic form is given by:

$$dp \wedge dq = \sum_{i<j} (y_i, y_j)\ dy_j \wedge dy_i$$

where $(y_i, y_j) = -(y_j, y_i)$ is the entry in row $i$ and column $j$ of the (anti-symmetric) matrix $A^tJA$. In particular, given a Hamiltonian, Hamilton's equations $(\dot{q} = \partial_p H, \dot{p} = -\partial_q H)$, in the new coordinates are:

$$\partial_{y_k} H = (y_1, y_k)\ \dot{y}_1 + ... + (y_{2n}, y_k)\ \dot{y}_{2n}, \quad k = 1, ..., 2n.$$

*proof:* For vectors $u, v \in \mathbb{R}^{2n}$ with base-point at $y \in \mathbb{R}^{2n}$ we have corresponding vectors $Au, Av \in \mathbb{R}^{2n}$ based at $x = \varphi(y)$, and $\omega(Au, Av) = Au \cdot JAv = u \cdot (A^tJAv)$. Thus, the matrix representation of $\omega$ in the standard basis of $\mathbb{R}^{2n}$ of vectors based at $y \in \mathbb{R}^{2n}$ is by the matrix $A^tJA$. Consider a Hamiltonian $H(x)$, and let $H(y) := H(\varphi(y))$. Then in the symplectic coordinates $x$, Hamilton's equations of motion are $\dot{x} = -J\nabla_x H$. By chain rule, since $x = \varphi(y)$, we have $\dot{x} = A\dot{y}$. On the other hand, $d_y H = d_{\varphi(y)} H d_y \varphi = d_x H\ A$. By the definition of gradient, $\nabla_y H \cdot u = d_x H(Au) = \nabla_x H \cdot Au = (A^t \nabla_x H) \cdot u$ for any $u \in \mathbb{R}^{2n}$, so that $\nabla_y H = A^t \nabla_x H$. Hence:

$$A\dot{y} = \dot{x} = -J\nabla_x H \Rightarrow JA\dot{y} = \nabla_x H \Rightarrow A^tJA\dot{y} = \nabla_y H.$$

□

The *Poisson brackets* associated to a general coordinate change $\varphi(y) = x$ arise in a similar manner as the entries of the inverse matrix $(A^tJA)^{-1}$. We have:

**Poisson brackets:** In coordinates $\varphi(y) = (q, p)$, let $\{y_i, y_j\} = -\{y_j, y_i\}$ be the entry in row $j$ and column $i$ of the (anti-symmetric) matrix $(A^tJA)^{-1}$. Then:

$$\{y_i, y_j\} = \omega(X_{y_j}, X_{y_i}),$$

and, given a Hamiltonian, Hamilton's equations $(\dot{q} = \partial_p H, \dot{p} = -\partial_q H)$, in the new coordinates are:

$$\dot{y}_k = \{y_k, y_1\}\ \partial_{y_1} H + ... + \{y_k, y_{2n}\}\ \partial_{y_{2n}} H, \quad k = 1, ..., 2n.$$

*proof:* Consider $\pi_{ij} := \omega(X_{y_j}, X_{y_i}) = dy_i(X_{y_j}) = -\pi_{ji}$ where $X_{y_j}$ is the Hamiltonian vector field of $y_j : \mathbb{R}^{2n} \to \mathbb{R}$. Then for a Hamiltonian, $H$, with Hamiltonian vector field $X_H$, we have:

$$\dot{y}_k = dy_k(X_H) = \omega(X_H, X_{y_k}) = -dH(X_{y_k}) = -\sum_j \partial_{y_j} H dy_j(X_{y_k}) = \sum_j \pi_{kj} \partial_{y_j} H$$

On the other hand, we have $\dot{y} = (A^tJA)^{-1}\nabla_y H = \sum_j \{y_k, y_j\}\ \partial_{y_j} H$ so that $\pi_{kj} = \{y_k, y_j\}$. □

Note that the conditions to be a symplectic change of coordinates, $A^tJA = J$, may be expressed as:

$$(y_i, y_{j+n}) = \{y_{j+n}, y_i\} = \delta_{ij}, \quad i, j \in \{1, ..., n\}.$$

Moreover, for two functions $f(q, p), g(q, p)$, we have the coordinate expression:

$$\{f, g\} = \omega(X_g, X_f) = df(X_g) = \sum_j (\partial_{p_j} g)\ (\partial_{q_j} f) - (\partial_{q_j} g)\ (\partial_{p_j} f).$$

The basic method in the Hamiltonian formalism is seeking 'good' changes of coordinates of a given system, in which properties of the Lagrange parentheses and Poisson brackets play a fundamental role. We will now return to some general considerations in perturbation theory, and then finish with a few examples.

According to Poincaré, the 'general problem of dynamics' consists in describing trajectories of a Hamiltonian system:

$$H_\varepsilon = H_o(I) + \varepsilon H(I,\theta), \quad \omega = dI \wedge d\theta$$

where $(I,\theta) \in \mathbb{R}^n \times T^n$ are symplectic coordinates and $\varepsilon$ is a small parameter.

When $\varepsilon = 0$, one has the 'unperturbed system', $H_o(I)$, which is in the Liouville integrable form and is solved:

$$I(t) \equiv I_o = cst., \quad \theta(t) = \theta_o + \omega(I_o)\ t$$

with $\omega(I) := \partial_I H_o \in \mathbb{R}^n$. Perturbation methods consist in determining conditions which guarantee the 'continuation' of orbits or persistence of properties of the understood $\varepsilon = 0$ system to the 'perturbed system' $H_\varepsilon$, at least for small, $0 < \varepsilon << 1$, parameter values. Seeking coordinate changes to convert $H_\varepsilon$ into a Liouville integrable form leads to:

**Birkhoff normal forms:** Consider $\omega_o := \omega(I_o)$ satisfying $\omega_o \cdot \vec{k} \neq 0$ for any $\vec{k} \in \mathbb{Z}^n$ (one says $\omega_o$ is a *non-resonant* frequency). Then for each $N \in \mathbb{N}$, there exists symplectic coordinates $(J,\varphi) \in \mathbb{R}^n \times T^n$ such that:

$$H_\varepsilon = \omega_o \cdot J + \varepsilon H_1(J) + ... + \varepsilon^{N-1} H_{N-1}(J) + \varepsilon^N H_N(J,\varphi).$$

See appendices 6,7 of Arnold. In short, small perturbations of integrable systems may be put into a form which is 'integrable upto order $\varepsilon^N$'. Attempting to take a limit as $N \to \infty$, the coordinate transformations in general become defined by divergent formal power series. The general idea in the construction of these series is to use a sequence of generating function to step by step 'kill' the terms of order $\varepsilon^k$. For example, let us consider:

$$H_\varepsilon = \omega_o \cdot I + \varepsilon H(I,\theta)$$

with $\omega_o$ non-resonant. We seek a symplectic transformation $\varphi_\varepsilon$, given as the time $\varepsilon$ flow of the Hamiltonian vector field $X_\chi$, to kill the terms of order $\varepsilon$. We have:

$$H_\varepsilon \circ \varphi_\varepsilon = H_\varepsilon + \varepsilon\{H_\varepsilon, \chi\} + O(\varepsilon^2) = \omega_o \cdot I + \varepsilon(H + \{\omega_o \cdot I, \chi\}) + O(\varepsilon^2).$$

By writing $H = \sum_{k\in\mathbb{Z}^n} h_k e^{ik\cdot\theta}, \chi = \sum_{k\in\mathbb{Z}^n} \chi_k e^{ik\cdot\theta}$ in Fourier series, we have:

$$\{\omega_o \cdot I, \chi\} = \partial_I \chi \cdot \partial_\theta(\omega_o \cdot I) - \partial_\theta \chi \cdot \omega_o = -\sum_{k\in\mathbb{Z}^n} (\omega_o \cdot k) i \chi_k e^{ik\cdot\theta}$$

$$\Rightarrow H_\varepsilon \circ \varphi_\varepsilon = \omega_o \cdot I + \varepsilon \left( \sum_{k\in\mathbb{Z}^n} (h_k - (\omega_o \cdot k) i \chi_k) e^{ik\cdot\theta} \right) + O(\varepsilon^2).$$

Since $\omega_o$ is non-resonant, we take $\chi_k := \frac{h_k}{i\omega_o \cdot k}, k \in \mathbb{Z}^n \backslash 0$, to obtain:

$$H_\varepsilon \circ \varphi_\varepsilon = \omega_o \cdot I + \varepsilon h_o(I) + O(\varepsilon^2).$$

One would then iterated this process to succesively 'kill' the $\theta$ dependence in terms of order $\varepsilon^k$. In fact when $H(I,\theta)$ is analytic, the decay rates of the fourier coefficients allow one to define $\chi_k$ at each stage by a finite trigonometric polynomial. As one takes $k \to \infty$, the polynomials $\chi_k$ tend to an infinite power series which –in general– is only a 'formal' power series, ie divergent.
KAM...implicit function theorem...RC3BP...forced pendulums...

## References

Our principal reference is the textbook of Arnold[1]. Some additional references:

- Feynman's lectures [2], are good for seeing more physical explanations of the concepts. Also the more elementary [3], has nice discussion.
- One may find various clever or 'fun' illustrations of mechanical principles in the books of Uspenskii, Levi, or articles of Tokieda [4]
- For more on calculus of variations, the book of Levi [5], is a nice supplement to Arnold's book. More focused on calculus of variations itself, one can see the books of Gelfand-Fomin, Young, or Moser [6].
- Meyer's notes [7] contain examples of Hamiltonian systems and perturbation methods. There is also a more expanded book [8].
- The book of Guckenheimer and Holmes [9] is a nice overview of dynamical systems. Also Arnold's texts [10] have many good insights.
- In addition to Arnold's text, there are a number of other general texts on mechanics [11].
- With any subject – once one feels comfortable or ambitious enough with the material – it is interesting practice to eventually study the motivations and developement of the subject. In mechanics, there are several interesting classical or historical works, eg [12] [13] to peruse.

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[1] V.I. Arnol'd, *Mathematical methods of classical mechanics*, (Vol. 60). Springer Science & Business Media, v. 60, (2013).

[2] R. Feynman, , R. Leighton, M. Sands, *The Feynman lectures on physics; vol. I.* American Journal of Physics 33.9 (1965). Available online here.

[3] I. Asimov, *Understanding Physics. 3 Vols.* Scientia, Rivista di Scienza 62.103 (1968).

[4] V.A. Uspenskiĭ, *Some applications of mechanics to mathematics.* Vol. 3. Blaisdell Publishing Company, (1961). M. Levi, *The mathematical mechanic. eg, T. Tokieda,* The hopping hoop. *The American mathematical monthly 104.2: 152-154 (1997). T. Tokieda,* Mechanical ideas in geometry. *The American mathematical monthly 105.8: 697-703 (1998). M. Gualtieri, T. Tadashi, L. Advis-Gaete, B. Carry, E. Reffet, C. Guthmann.* Golfer's dilemma. *American journal of physics 74, no. 6: 497-501 (2006).* Princeton University Press, (2009).

[5] M. Levi, *Classical mechanics with calculus of variations and optimal control: an intuitive introduction.* Vol. 69. American Mathematical Soc., (2014).

[6] I.M. Gelfand, S.V. Fomin, *Calculus of Variations Prentice-Hall.* Inc., Englewood Cliffs (1963). L. Young, *Lectures on the calculus of variations and optimal control theory.* Vol. 304. American Mathematical Soc., (2000). J. Moser, *Selected chapters in the calculus of variations.* Springer Science & Business Media, (2003).

[7] K. Meyer, *Periodic solutions of the N-body problem.* Vol. 1719. Springer Science & Business Media, (1999).

[8] K. Meyer, D. Offin. *Hamiltonian systems – Introduction to Hamiltonian Dynamical Systems and the N-Body Problem.* Springer, Cham, 29-60, (2017).

[9] J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Vol. 42. Springer Science & Business Media, (2013).

[10] V.I. Arnold, *Ordinary Differential Equations*, MIT press, 1st edition, (1978). (I prefer the first (green and purple) edition.) V.I. Arnold, *Lectures on partial differential equations.* Moscow, (2004). V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations.* Vol. 250. Springer Science & Business Media, (2012).

[11] eg: M. Spivak, *Physics for mathematicians: Mechanics I.* Publish or Perish, (2010). Also the notes here. J.M. Souriau, *Structure of dynamical systems: a symplectic view of physics.* Vol. 149. Springer Science & Business Media, (2012). L. Landau, E. M. Lifshitz, *Mechanics, vol. I of course on theoretical physics.* American Journal of Physics 40.7 (1972).

[12] eg T. Heath, T. Little (trans.), *The works of Archimedes,* Courier Corporation, (2002). G. Galilei, *Dialogue Concerning the Two Chief World Systems*, (1632). G. Galilei, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, (1638). I. Newton, *Mathematical Principles of Natural Philosophy*, (1687). L. Euler, *Mechanica*, (1736). J. d'Alembert, *Traite de dynamique*, (1743). J. Lagrange, *Mécanique analytique*, (1788). G. Monge, *Traité élémentaire de statique,* 1810. S. Poisson, *Traité de mécanique. 2 Vols.*, (1838). W. Hamilton, *On a General Method of Expressing the Paths of Light, & of the Planets, by the Coefficients of a Characteristic Function,* (1833). W. Hamilton, *On a general method in dynamics. 2 essays,* (1834-35). C. Jacobi, *Vorlesungen uber dynamik.* (1884). L. Poinsot, *Eléments de statique,* (1861). L. Poinsot, *Théorie nouvelle de la rotation des corps,* (1851). P. Appell, *Traité de mécanique rationnelle. 4 Vols.*, (1911).

[13] more astronomy oriented: C. Ptolemy, *Almagest*, (∼100-170 a.d.). N. Copernicus, *On the Revolutions of the Heavenly Spheres*, (1543). J. Kepler, *Astronomia nova*, (1609). P. Laplace, *Traité de mécanique céleste (5 vols.)*, (1799). F. Tisserand, H. Andoyer, *Leçons de cosmographie*, (1912). F. Tisserand, *Traité de mécanique céleste (4 vols.)*, (1889). H. Poincaré, *Cours d'astronomie générale*, (1906). H. Poincaré, *Leçons de mécanique céleste (3 vols.)*, (1905). H. Poincaré, *Les méthodes nouvelles de la mécanique céleste (3 vols.)*, (1892).