

Some theorems from vector calculus

In our study of electromagnetic fields, we will apply the following results.

POINCARÉ LEMMA: this gives a converse to the results that $\nabla \times (\nabla f) = \nabla \cdot (\nabla \times X) = 0$.

1. Let X be a vector field on \mathbb{R}^3 with $\nabla \times X = 0$. Then there exists a function f with $X = \nabla f$.
2. Let X be a vector field on \mathbb{R}^3 with $\nabla \cdot X = 0$. Then there exists a vector field Y with $X = \nabla \times Y$.

proof: 1. Fix a ‘base-point’ x_o and define $f(x) := \int_C X \cdot d\vec{s}$ where C is a curve from x_o to x . Then f is a well-defined function since for C' from x_o to x we have $\int_C X \cdot d\vec{s} - \int_{C'} X \cdot d\vec{s} = \int_{\partial\Sigma} X \cdot d\vec{s} = \int_{\Sigma} (\nabla \times X) \cdot d\vec{S} = 0$ for some surface Σ with $\partial\Sigma = C - C'$. Note that $f(x_o) = 0$ and so for *any* curve C we have $\int_C \nabla f \cdot d\vec{s} = \int_C X \cdot d\vec{s}$, hence $\nabla f = X$.

2. We may take $Y(x) := \int_0^1 tX(tx) \times x dt$ and compute that $\nabla \times Y = X$ (using $\nabla \cdot X = 0$). □

Note that the solutions are not unique, eg in case 1 we may shift $f \mapsto f + c$ by an arbitrary constant and in case 2 shift $Y \mapsto Y + \nabla f$ by addition of an arbitrary gradient vector field. Moreover the results can be shown to hold for regions where any closed curve is the boundary of some surface (for 1) or any closed surface is the boundary of some region (for 2).

HELMHOLTZ DECOMPOSITION: Let $\Omega \subset \mathbb{R}^3$ be a compact region with smooth boundary $\partial\Omega$. Then a vector field X on Ω may be written as:

$$X = -\nabla f + \nabla \times Y$$

for some function f and vector field Y on Ω .

proof: For $x_o \in \Omega$ let $r : \Omega \rightarrow \mathbb{R}$, $x \mapsto |x - x_o| = r$. Then $\Delta \frac{1}{r} = -4\pi\delta(x - x_o)$ so that:

$$-4\pi X(x_o) = \int_{x \in \Omega} \Delta \left(\frac{1}{r} \right) X(x) dV = \Delta^\circ \int_{x \in \Omega} \frac{X(x)}{|x - x_o|} dV$$

where on the right, Δ° is the vector Laplacian wrt x_o . Applying the definition of the vector Laplacian, we have:

$$-4\pi X(x_o) = \nabla^\circ \int_{x \in \Omega} \nabla^\circ \left(\frac{1}{r} \right) \cdot X(x) dV - \nabla^\circ \times \left(\int_{x \in \Omega} \nabla^\circ \left(\frac{1}{r} \right) \times X(x) dV \right)$$

by using the identities $\nabla \cdot (gV) = \nabla g \cdot V + g\nabla \cdot V$ and $\nabla \times (gV) = \nabla g \times V + g\nabla \times V$. Hence we have our decomposition by taking:

$$4\pi f(x_o) = \int_{x \in \Omega} \nabla^\circ \left(\frac{1}{r} \right) \cdot X(x) dV, \quad 4\pi Y(x_o) = \int_{x \in \Omega} \nabla^\circ \left(\frac{1}{r} \right) \times X(x) dV$$

where the gradients, ∇° , are wrt x_o . Switching to gradients wrt x , $\nabla \frac{1}{r} = -\nabla^\circ \frac{1}{r}$, and integrating by parts with the divergence theorem, we may rearrange as:

$$(*) \quad f(x_o) = \frac{1}{4\pi} \left(\int_{x \in \Omega} \frac{\nabla \cdot X(x) dV}{|x - x_o|} - \int_{y \in \partial\Omega} \frac{X(y) \cdot \nu(y) dA}{|y - x_o|} \right)$$

$$(**) \quad Y(x_o) = \frac{1}{4\pi} \left(\int_{x \in \Omega} \frac{\nabla \times X(x) dV}{|x - x_o|} - \int_{y \in \partial\Omega} \frac{\nu(y) \times X(y) dA}{|y - x_o|} \right)$$

where ν is the outward normal to $\partial\Omega$. □

From the explicit formulas (*), (**), in the derivation of the Helmholtz decomposition, we obtain the useful corollaries:

1. If X is a vector field on \mathbb{R}^3 such that $\lim_{r \rightarrow \infty} rX(\vec{r}) = 0$ then $X = -\nabla f + \nabla \times Y$ with

$$f(x_o) = \frac{1}{4\pi} \int_{x \in \mathbb{R}^3} \frac{\nabla \cdot X(x)}{|x - x_o|} dV, \quad Y(x_o) = \frac{1}{4\pi} \int_{x \in \mathbb{R}^3} \frac{\nabla \times X(x)}{|x - x_o|} dV,$$

2. Given a function f and divergence free vector field Y with $\lim_{r \rightarrow \infty} r^2 f = \lim_{r \rightarrow \infty} r^2 Y = 0$ then there exists a vector field X with

$$\nabla \cdot X = f, \quad \nabla \times X = Y.$$

Moreover, if X satisfies the additional condition that $\lim_{r \rightarrow \infty} rX = 0$ then X is unique.

In our studies of static electricity and magnetism, we meet exactly the sort of equations in corollary (2) for which the ‘conditions at infinity’ are natural, as for physically realistic situations one is interested in the effects produced by bounded sources where far away the resulting fields should be small.