Midterm

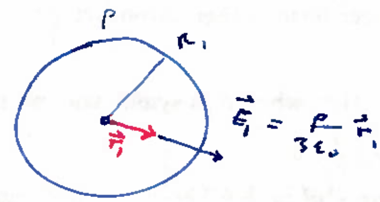
1. we first determine the electric field inside a homogeneous solid ball.

By spherical symmetry, the electric field is radial.
Consider a solid ball of radius $r < R_1$. By Gauss:

$$\int_{S_r^2} \vec{E} \cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0} = E(r)\, 4\pi r^2$$

$\Rightarrow \quad E(r) = \dfrac{\rho r}{3\varepsilon_0}$ where $E(r)$ is the electric field strength at distance r from the balls center.

Hence $\vec{E}_1 = \dfrac{\rho}{3\varepsilon_0}\vec{r}_1$ where $\vec{r}_1$ is the position vector from the ball's center:

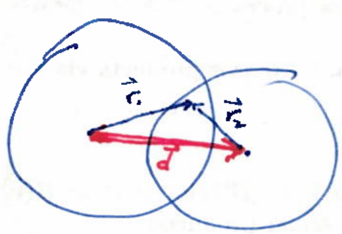


Likewise $\vec{E}_2 = -\dfrac{\rho}{3\varepsilon_0}\vec{r}_2$.

By superposition, at a point in the spheres intersection:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho}{3\varepsilon_0}(\vec{r}_1 - \vec{r}_2) = \frac{\rho}{3\varepsilon_0}\vec{d}$$

where $\vec{d}$ is the vector from the center of the spheres (constant).



2. The potential φ generating the electric field is constant over Ω:

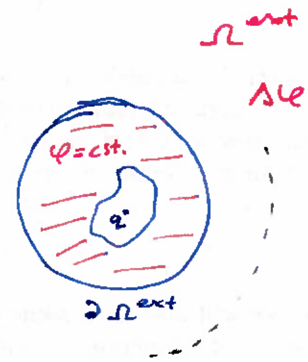
$$\varphi|_{\Omega} = \text{cst.}$$

Exterior to Ω there are no charges, so

$$\Delta \varphi = 0 \quad \text{in} \quad \Omega^{ext}$$

As the exterior Dirichlet problem has a unique solution, and $\partial \Omega^{ext}$ is a sphere, φ has the form:

$$\varphi(r) = \frac{a}{r} \quad \text{for } a \text{ constant.}$$

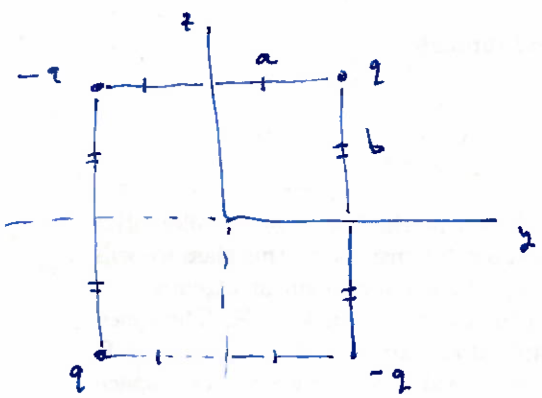


hence $\vec{E}^{ext} = -\nabla\left(\frac{a}{r}\right) = \frac{a}{r^2}\vec{r}$

and by ~~the~~ Gauss applied to a concentric sphere we obtain:

$$\vec{E}^{ext} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{r}$$, since $Q_{int} = q$ (the conductor has no charge).

3. We use the method of images with the following configuration:



the field produced by these 4 charges is perpendicular to $\partial\Omega$, so by uniqueness is the resulting electric field in Ω^c (in Ω^{int} $\vec{E} = 0$ by def. of conductor).

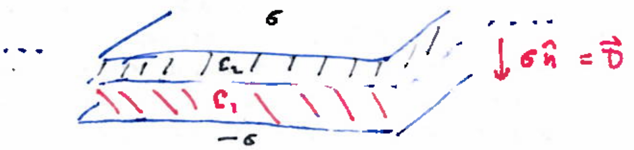
Letting charge q be at coordinates $y = a$, $z = b$ (and $x = 0$) we have (superposition)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\frac{(x, y-a, z-b)}{(x^2+(y-a)^2+(z-b)^2)^{3/2}} - \frac{(x, y-a, z+b)}{(x^2+(y-a)^2+(z+b)^2)^{3/2}} + \frac{(x, y+a, z+b)}{(x^2+(y+a)^2+(z+b)^2)^{3/2}} - \frac{(x, y+a, z-b)}{(x^2+(y+a)^2+(z-b)^2)^{3/2}}\right) \quad \text{in } \Omega^c.$$

The total charge resulting on Ω is $\underline{q}$.

4. For parallel plate capacitors we use the approximation by an INFINITE PLANES (ignore edge effects).

We first determine the displacement field $\vec{D}$ for infinite planes.



by symmetry, $\vec{D}$ is vertical. From $\nabla\cdot\vec{D} = \rho_0$ we deduce

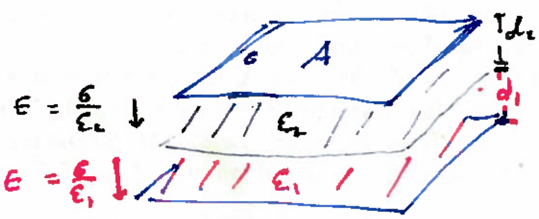
$$\vec{D} = \sigma\hat{n}$$

between the plates.

The electric field between the plates is then given precisewise:

$$\vec{E} = \begin{cases} \frac{\sigma}{\varepsilon_2}\hat{n} \\ \frac{\sigma}{\varepsilon_1}\hat{n} \end{cases}.$$

Now, for the parallel plate capacitor:



we have total charge $Q = \sigma A$ and $\Delta V = \frac{\sigma d_2}{\varepsilon_2} + \frac{\sigma d_1}{\varepsilon_1}$ so that:

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{\sigma\left(\frac{d_2}{\varepsilon_2} + \frac{d_1}{\varepsilon_1}\right)} = \frac{A\varepsilon_1\varepsilon_2}{d_2\varepsilon_1 + d_1\varepsilon_2}.$$

5. One may consider $\vec{E}_0 = -\nabla\varphi_0$ and the field $\vec{E}_0$ is generated by the distribution $\sigma_0 = -\varepsilon_0\,\partial_\nu\varphi_0^+$ on $\partial\Omega$ [and $\vec{E}_0 = 0$ in Ω].

when Ω^c is filled with a material of permittivity ε, then

$$\vec{D} = \begin{cases} 0 & \text{on } \Omega \\ \varepsilon\vec{E}_0 & \text{on } \Omega^c \end{cases} \quad \text{and} \quad \vec{D}_+\cdot\nu = \varepsilon\vec{E}\cdot\nu = \sigma_0 = -\varepsilon_0\partial_\nu\varphi_0^+$$
$$= \varepsilon_0\vec{E}_0\cdot\nu.$$

hence $\vec{E} = \frac{\varepsilon_0}{\varepsilon}\vec{E}_0$ on $\partial\Omega$, and by uniqueness of solutions $\Delta\varphi = 0$ on Ω^c ($\varphi \to 0$ @ ∞) w/ $\varphi = \frac{\varepsilon_0}{\varepsilon}\varphi_0$ we have $\vec{E} = -\nabla\varphi$ on Ω^c is related to $\vec{E}_0$ by:

$$\vec{E} = \frac{\varepsilon_0}{\varepsilon}\vec{E}_0$$

* alternately, $\vec{E}_0 = -\nabla \varphi_0$ on Ω^c is determined uniquely by the φ_0 solving $\Delta \varphi_0 = 0$ ($\varphi_0 \to 0$ @ ∞) in Ω^c with

$$\varepsilon_0 \partial_\nu \varphi_0 = -\sigma_0 \quad \text{giving } \Omega \text{ charge } Q.$$

The field $\vec{E}$ over Ω^c is determined uniquely by the potential φ solving $\Delta \varphi = 0$ ($\varphi \to 0$ @ ∞) in Ω^c with

$\varepsilon \partial_\nu \varphi = -\sigma_0$ [the charge Q on Ω is unchanged]

Hence $\varphi = \frac{\varepsilon_0}{\varepsilon} \varphi_0$ and so $\vec{E} = -\nabla \varphi = \frac{\varepsilon_0}{\varepsilon} \vec{E}_0$ in Ω^c

[both fields vanish in Ω].