

HILL'S EQUATION.¹

The 2nd order o.d.e.'s

$$\frac{d^2y}{dx^2} + p(x)y = 0 \tag{1}$$

have a central place in dynamics. As we have seen, any regular 2nd order o.d.e. can be converted to the form (1) by an explicit change of variables. Of particular interest is when the coefficient $p(x)$ is some periodic function (*Hill equation*). For example when we study the variational equations around some periodic orbit, in a 3-dimensional space, the behaviour of nearby solutions is often governed by solutions to an appropriate Hill equation.

In this note we will mention some generalities (§1, 2) which lead to the study of Hill equations. In §3 we will consider an explicit example (the *Mathieu equation*: eq. (3)) where we are able to apply the method of series underlying a stability diagram (§3.2) for this system ('Arnold tongues').

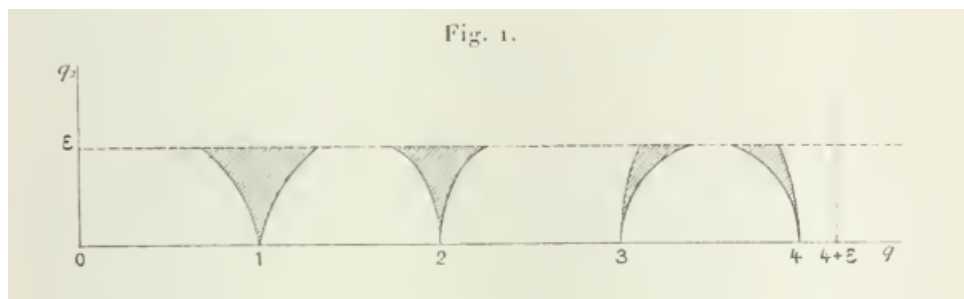


Figure 1: The 'Arnold tongues' for the Mathieu equation: from §179 of H. Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*, T. II, pg. 243.

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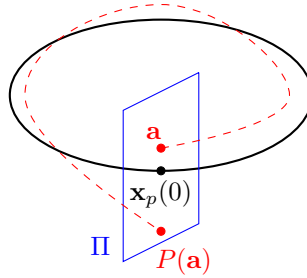
1 Poincaré return map, monodromy

As an initial seed, suppose we have some periodic solution $\mathbf{x}_p(t+T) = \mathbf{x}_p(t)$ to some system of ode's $\mathbf{x}' = \mathbf{v}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$.

We would like to study solutions close to this periodic solution. Choose some plane, Π , through $\mathbf{x}_p(0)$ transverse to the tangent $\mathbf{x}'_p(0)$. Consider a solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{a} \in \Pi$ in this plane. If $\mathbf{a}$ is sufficiently close to $\mathbf{x}_p(0)$ then, after some time T' (close to T), the solution $\mathbf{x}(t)$ will return to intersect Π at some point $P(\mathbf{a}) = \mathbf{x}(T')$. This *Poincaré first return map* to Π we denote:

$$P : \Pi \dashrightarrow \Pi.$$

It need not be defined on all of Π (there is no reason initial conditions in Π far from $\mathbf{x}_p(0)$ need to ever return to Π again!). But, at least P 's domain includes some open neighborhood of $\mathbf{x}_p(0) \in \Pi$. For example, we always know $P(\mathbf{x}_p(0)) = \mathbf{x}_p(0)$ is a fixed point of the Poincaré map.



Computing some explicit formula for Poincaré maps in general is a hopeless task (one would require explicit formulas for general flows). Instead, in general, one hopes to extract some information about properties or approximations to the Poincaré map. In this note our aim will be to extract information approximating the *linearization* of the Poincaré map:

Definition 1. For a periodic orbit $\mathbf{x}_p(t) = \mathbf{x}_p(t+T)$ of some system of ode's $\mathbf{x}' = \mathbf{v}(\mathbf{x})$, with Poincaré (first return) map $P : \Pi \dashrightarrow \Pi$ from some plane $\Pi \ni \mathbf{x}_p(0)$ transverse to $\mathbf{x}'_p(0)$, then the monodromy mapping of the periodic orbit is the linearization, at $\mathbf{x}_p(0)$, of its Poincaré map:

$$M := d_{\mathbf{x}_p(0)} P : \Pi \rightarrow \Pi.$$

Understanding this monodromy mapping amounts to understanding the transport along solutions to the associated variational equations along the periodic orbit:

Proposition 1. The monodromy mapping along a periodic orbit is determined through:

$$M\xi_o \equiv \xi(T) \mod \mathbf{x}'_p(0)$$

for $\xi(t)$ with $\xi(0) = \xi_o$ a solution to the variational equations (linearization) along the periodic orbit: $\xi' = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}|_{\mathbf{x}_p(t)} \xi$, $\xi(0) = \xi_o$.

Proof. For simplicity, let us translate $\mathbf{x}_p(0)$ to the origin. Consider some vector $\xi_o \in \Pi$. Then, by definition, $M\xi_o = \frac{d}{d\varepsilon}|_{\varepsilon=0}P(\varepsilon\xi_o)$. As well, by definition, $P(\varepsilon\xi_o) = \mathbf{x}_\varepsilon(T(\varepsilon))$, where $\mathbf{x}_\varepsilon(t)$ is a solution with initial condition $\mathbf{x}_\varepsilon(0) = \varepsilon\xi_o$, and which intersects again Π after time $T(\varepsilon)$ (tending to T as $\varepsilon \rightarrow 0$). Writing $\mathbf{x}_\varepsilon(t) = \mathbf{x}_p(t) + \varepsilon\xi(t, \varepsilon)$ we have $\xi' = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}|_{\mathbf{x}_p(t)}\xi + O(\varepsilon)$ and:

$$M\xi_o = \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathbf{x}_\varepsilon(T(\varepsilon)) = \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathbf{x}_p(T(\varepsilon)) + \varepsilon\xi(T(\varepsilon), \varepsilon) = T'(0)\mathbf{x}'_p(0) + \xi(T, 0)$$

where $\xi(t, 0)$ is a solution to the variational equations around our periodic orbit with initial condition $\xi(0, 0) = \xi_o$. $\square$

To obtain the variational equations around this periodic orbit we may choose, at each time t , a basis completing the tangent direction $\mathbf{x}'_p(t) = \mathbf{x}'_p(t+T)$ to have some periodic basis² $\mathbf{u}_1(t) = \mathbf{u}_1(t+T)$, $\mathbf{u}_2(t) = \mathbf{u}_2(t+T)$ along the periodic orbit. We use this moving basis to examine nearby points to this periodic orbit through the coordinates

$$(\tau, v_1, v_2) \mapsto \mathbf{x}_p(\tau) + v_1\mathbf{u}_1(\tau) + v_2\mathbf{u}_2(\tau).$$

In these coordinates, a nearby solution to our periodic solution $(\tau(t), \varepsilon v_1(t), \varepsilon v_2(t))$ (for $\varepsilon \ll 1$ small) satisfies equations of motion of the form³:

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + O(\varepsilon), \quad \tau' = 1 + O(\varepsilon).$$

Sending $\varepsilon \rightarrow 0$ we have explicit variational equations. Examining the monodromy along the periodic orbit amounts to examining the transport map

$$(v_1^o, v_2^o) \mapsto (v_1(T), v_2(T))$$

for $v_j(t)$ solving $\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with initial conditions $v_j^o = v_j(0)$.

Remark. Recall a 2×2 system $\mathbf{x}' = A(t)\mathbf{x}$, where $A(t) \neq \sigma(t)I$ is not a multiple of the identity, can be transformed to a 2nd order ode in normal form (1).

2 Perturbative Hill equation

We will consider a Hill equation in the particular form:

$$\frac{d^2x}{dt^2} + x = \varepsilon f(t)x \tag{2}$$

²That such a choice exists, follows by considering that the tangent lines along the periodic orbit trace out some surface in $\mathbb{R}^3$. For a pair of distinct points P, Q not lying on this surface (and with the line PQ not intersecting our orbit) the vectors $\mathbf{u}_1(t) = P - \mathbf{x}_p(t)$ and $\mathbf{u}_2(t) = Q - \mathbf{x}_p(t)$ are independent and transverse to the periodic orbit.

³Explicitly, set $\frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ for the Jacobian of $\mathbf{v}$ evaluated at $\mathbf{x}_p(t)$. Then: $a = \frac{\mathbf{x}'_p \wedge (\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{u}_1 - \mathbf{u}'_1) \wedge \mathbf{u}_2}{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2}$, $b = \frac{\mathbf{x}'_p \wedge (\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{u}_2 - \mathbf{u}'_2) \wedge \mathbf{u}_1}{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2}$, and, $c = \frac{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge (\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{u}_1 - \mathbf{u}'_1)}{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2}$, $d = \frac{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge (\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{u}_2 - \mathbf{u}'_2)}{\mathbf{x}'_p \wedge \mathbf{u}_1 \wedge \mathbf{u}_2}$.

where $\varepsilon \in \mathbb{R}$ is a (small) parameter and $f(t+T) = f(t)$ a function of period $T > 0$. A perturbative situation such as this might arise from the following considerations:

- Consider some simple physical system (1 degree of freedom) modelled by a 2nd order ode

$$(*) \quad x'' = F(x)$$

(for example a pendulum/swing). We suppose this system has some stable equilibrium configuration, say at $x = 0$ (so $F(0) = 0, F'(0) < 0$).

- We ask how one might ‘excite’ the system $(*)$ by applying adequately chosen (small) external forces to the system. We will aim to choose these external forces (perturbations) on the system in a periodic (repeating) manner so that the equilibrium $x = 0$ becomes unstable. For instance, if we are sitting at the bottom of the swing, we may move our body around in a repeating motion to slightly perturb the situation. So we will consider:

$$(*)' \quad x'' = F(x, t, \varepsilon) = F(x, t+T, \varepsilon)$$

where the dependence on the (small) parameter ε and (periodic) dependence on time t capture our applied external forces described above.

- This new system $(*)'$ has:

$$F(x, t, 0) = F(x), \quad F(0, t, \varepsilon) = F(0) = 0$$

since we consider $x = 0$ to remain centered at the equilibrium configuration (and ε to represent the strength of our applied forces). Taking a Taylor expansion, we have the system $(*)'$ as

$$x'' = F_1(t, \varepsilon)x + O(x^2)$$

where $F_1(t, \varepsilon) = -c^2 + \varepsilon f_1(t) + O(\varepsilon^2)$ (for $c^2 = -F'(0)$). Being concerned with the behaviour near the equilibrium position ($x = 0$) and small excitations ($|\varepsilon| \ll 1$), we have leading order terms: $x'' = (-c^2 + \varepsilon f_1(t))x + o(\varepsilon, x)$, which upon re-scaling the time $\tau = ct$ and truncating is just a Hill’s equation in the perturbation form (2).

Applying the method of series to the perturbative Hill equation (2)

Theorem 1. *The transport map*

$$\mathcal{T}_{t, \varepsilon} : \begin{pmatrix} x_o \\ x'_o \end{pmatrix} \mapsto \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

for $x(t)$ a solution to (2) having initial conditions $x(0) = x_o, x'(0) = x'_o$ has

1. $\det \mathcal{T}_{t, \varepsilon} = 1$,
2. $\text{tr} \mathcal{T}_{t, \varepsilon} = 2 \cos t + \varepsilon \sin t \int_0^t f(\tau) d\tau + \varepsilon^2 \tau_2(t) + \dots$

where, for the trace, the 2nd order coefficient is, explicitly:

$$\tau_2(t) = \int_0^t \int_0^\tau f(\sigma)f(\tau) \sin(\tau - \sigma) \sin(t + \sigma - \tau) d\sigma d\tau$$

Proof. Let us first note that the determinant of the transport map is, for a general eq. (1) always one (Wronskian formula). To find a series expansion, $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$, in ε of a general solution to (2) we may determine by a series of iterated quadratures. Namely we use method of variation of constants to solve succesively:

$$x_0'' + x_0 = 0, \quad x_1'' + x_1 = f x_0, \quad \dots, \quad x_{k+1}'' + x_{k+1} = f x_k$$

and find, given an initial $x_0(t) = x_o \cos t + x_o' \sin t$, we have iteratively

$$x_{k+1}(t) = \int_0^t f(\tau) x_k(\tau) \sin(t - \tau) d\tau$$

for the higher order terms. Note: $x_{k+1}'(t) = \int_0^t f(\tau) x_k(\tau) \cos(t - \tau) d\tau$. Thus we can compute (upto quadratures) the transport map itself to any order in ε via the two recursive series:

$$\xi_0(t) = \cos t, \quad \xi_{k+1}(t) = \int_0^t f(\tau) \xi_k(\tau) \sin(t - \tau) d\tau$$

$$\eta_0(t) = \sin t, \quad \eta_{k+1}(t) = \int_0^t f(\tau) \eta_k(\tau) \sin(t - \tau) d\tau$$

with corresponding solutions $\xi(t) = \xi_0(t) + \varepsilon \xi_1(t) + \varepsilon^2 \xi_2(t) + \dots$ and $\eta(t) = \eta_0(t) + \varepsilon \eta_1(t) + \varepsilon^2 \eta_2(t) + \dots$ and the transport map

$$\mathcal{T}_{t,\varepsilon} \begin{pmatrix} x_o \\ x_o' \end{pmatrix} = \begin{pmatrix} \xi(t) & \eta(t) \\ \xi'(t) & \eta'(t) \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}.$$

In particular, for its trace, we have:

$$\xi(t) + \eta'(t) = 2 \cos t + \sum_{k=1}^{\infty} \varepsilon^k \int_0^t f(\tau) (\xi_{k-1}(\tau) \sin(t - \tau) + \eta_{k-1}(\tau) \cos(t - \tau)) d\tau$$

and we used just this recursion to find the first two terms announced above. $\square$

Remark. So in principle we can compute the transport map of (2) (i.e. its flow/general solution) to any desired order in ε (upto quadratures). In the present situation, the transport being by linear maps on the plane, the most basic information (the eigenvalues) follow from the trace and determinant.

Remark. Regarding our motivating scenario (to try to excite a stable equilibrium), relevant information is to examine how the system has changed after one period of our applied forcing, that is we are interested in the properties of the ‘snapshots’ we take with the monodromy $M = \mathcal{T}_{T,\varepsilon}$.

3 Mathieu equation

A Hill equation in the particular form:

$$\frac{d^2 x}{dt^2} + x = \varepsilon \cos(\omega t)x \quad (3)$$

(for parameters $\varepsilon, \omega \in \mathbb{R}$) is known as the *Mathieu equation*. The perturbing coefficient $\varepsilon \cos(\omega t)$ has associated the natural period

$$T = 2\pi/\omega.$$

Remark. *The Mathieu equation is a special case of the perturbed Hill equation (2). It arises from considering in (2) a Fourier expansion of $f(t) = f_o + \sum (a_n \cos n\omega t + b_n \sin n\omega t)$ and truncating at the first order harmonics to have $x'' + (1 - \varepsilon f_o)x = \varepsilon(a_1 \cos \omega t + b_1 \sin \omega t)x = \varepsilon A \cos \omega(t + t_o)x$, which, upon re-scaling the time variable $\tau = \sqrt{1 - \varepsilon f_o}(t + t_o)$ (and re-labeling parameters) is just the form of a Mathieu equation.*

3.1 Expansion of Monodromy

For the Mathieu equation we can be more explicit in our above expansions (theorem 1)

Theorem 2. *The monodromy map*

$$M : \begin{pmatrix} x_o \\ x'_o \end{pmatrix} \mapsto \begin{pmatrix} x(T) \\ x'(T) \end{pmatrix}$$

for $x(t)$ a solution to the Mathieu equation (3) with initial conditions $x(0) = x_o, x'(0) = x'_o$ has determinant one ($\det M = 1$) and trace

$$\text{tr}(M) = 2 \cos T + \varepsilon^2 \frac{T^3 \sin T}{8(T^2 - \pi^2)} + \dots \quad (4)$$

Proof. We use the general expansion in theorem 1 for our particular case of Mathieu's equation (3): $f(t) = \cos \omega t$ (for $\omega = 2\pi/T$). The first coefficient (of ε) vanishes since we have zero average:

$$\int_0^{2\pi/\omega} \cos(\omega t) dt = 0.$$

For the 2nd order coefficient (of ε^2) we have some trig integrals to consider:

$$\tau_2(T) = \int_0^T \cos(\omega \tau) \int_0^\tau \cos(\omega \sigma) \sin(\tau - \sigma) \sin(T + \sigma - \tau) d\sigma d\tau$$

For these we can recall the product/sum formulas:

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y), \quad 2 \cos x \cos y = \cos(x - y) + \cos(x + y),$$

Then our first integrand (with respect to σ) is

$$\frac{1}{2} \int_0^\tau \cos(\omega\sigma) \cos(2\tau - 2\sigma - T) - \cos(\omega\sigma) \cos T \, d\sigma$$

where the first term (with product/sum again) integrates to:

$$\frac{1}{4} \int_0^\tau \cos((\omega-2)\sigma + 2\tau - T) + \cos((\omega+2)\sigma - 2\tau + T) d\sigma = \frac{\sin(2\tau - T) + \sin T \cos(\omega\tau) - \frac{1}{2}\omega \cos T \sin(\omega\tau)}{4 - \omega^2}$$

while the 2nd to $-\frac{\cos T \sin(\omega\tau)}{2\omega}$. To finish the job, we recall $\int_0^T \cos(\omega\tau) \sin(\omega\tau) d\tau = 0$, $\int_0^T \cos^2(\omega\tau) d\tau = \frac{T}{2}$ so that

$$\tau_2 = \frac{T \sin T}{2(4 - \omega^2)} + \frac{1}{4 - \omega^2} \int_0^T \cos(\omega\tau) \sin(2\tau - T) d\tau$$

and, again using the product/sum formula: $2 \sin x \cos y = \sin(x+y) + \sin(x-y)$, we find the last integral vanishes (exercise!). So, expressed in terms of the period $T = 2\pi/\omega$, our 2nd order coefficient is $\frac{T \sin T}{2(4 - \omega^2)} = \frac{T^3 \sin T}{8(T^2 - \pi^2)}$. $\square$

Remark. *With different notation for the parameters ($T = \pi q, \varepsilon = q_1/q^2$) we find these expansions (upto order ε^4) for the trace of the monodromy of the Mathieu equation in §179 of H. Poincaré's Les Méthodes Nouvelles de la Mécanique Céleste, T. II, pg. 234. Poincaré attributes the method to F. Tisserand (Traite de Mécanique Céleste). Tisserand was studying such expansions to obtain more precise descriptions for the motion of the moon, applying the following observations to compute expansions with the Mathieu equation more efficiently: (1°) the trace will be an even function in ε (so its expansion in ε will contain only even powers), as well, in the notation of theorem 1, one finds (2°) $\frac{1}{2} \text{tr}(M) = \xi(T) = \eta'(T)$, so it suffices to compute just one of these series.*

3.2 Arnold tongues (stability)

As a consequence of our expansion computations we will deduce some stability properties of the Mathieu equation (when $|\varepsilon| \ll 1$ is small).

To take successive snapshots of a solution to (3) every time interval $T = 2\pi/\omega$: $\mathbf{x}(0), \mathbf{x}(T), \mathbf{x}(2T), \dots$, we apply the monodromy mapping M repeatedly to the appropriate initial condition:

$$\mathbf{x}_o \rightarrow M\mathbf{x}_o \rightarrow M^2\mathbf{x}_o \rightarrow \dots \rightarrow M^k\mathbf{x}_o \rightarrow \dots$$

the behaviour of such iterates is governed by the eigenvalues of the linear map $M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Namely:

- If M has an eigenvalue λ with $|\lambda| > 1$ then we have iterates $M^k\mathbf{x}_o$ escaping to infinity as $k \rightarrow \infty$.

- If M has an eigenvalue λ with $|\lambda| < 1$ then we have iterates with $M^k \mathbf{x}_o$ collapsing to the origin as $k \rightarrow \infty$.
- If M has a unit complex eigenvalue $|\lambda| = 1$ (so, being a real matrix, two complex conjugate eigenvalues) then M acts as a rotation: the iterates $M^k \mathbf{x}_o$ progress along circles centered about the origin.

There is also the exceptional case we may have a Jordan block with eigenvalues ± 1 , in which case there is also a certain instability in the sense that most solutions escape to infinity (due to the ‘shifting term’ in the Jordan block).

In our present situation, since $\det M = 1$:

- when $\det M = 1$, then the eigenvalues of M (roots of $\lambda^2 - \text{tr}(M)\lambda + 1 = 0$) are determined by the trace of M as
 - if $|\text{tr}(M)| < 2$, then the eigenvalues are both unit complex numbers: $\lambda = e^{i\theta}, \bar{\lambda} = e^{-i\theta}$. The iterations of M are by succesively rotating by some fixed angle.
 - if $|\text{tr}(M)| = 2$, then the eigenvalue of M is repeated ± 1 . To decide completely slightly more information is required, namely if there is a Jordan block or not.
 - if $|\text{tr}(M)| > 2$, then M has a pair of distinct real eigenvalues with $\lambda_1 \lambda_2 = 1$. Say $|\lambda_1| < 1$ and $|\lambda_2| = 1/|\lambda_1| > 1$, so that we see a hyperbolic behaviour for the iterates of M (the eigenspace of λ_1 being attractive and of λ_2 being repulsive).

These type of snapshots represent a *stable* situation: no matter how many snapshots we take, we do not see an initial condition (close to the origin) ever deviating substantially from the origin.

These type of snapshots represent an *unstable* situation: unless we happen to lie *exactly* along the eigenspace (line) corresponding to the contracting eigenvalue, then we will see a typical initial condition tending off to infinity after a few snapshots.

The unstable type of behaviour in the last item ($|\text{tr}(M)| > 2$) is what we hope to induce by an excitation of the system. It is just what we do when we are kids on the swing (to get the swing to move higher and higher).

Using an expansion (4), we can begin to describe that region in the T, ε -plane where $|\text{tr}(M)| > 2$ (at least close to the T -axis). For instance, along the T -axis (when $\varepsilon = 0$) the trace is $2 \cos T$ and so whenever $T \neq n\pi$ is not some multiple of π we will be in the stable situation (by continuity, will still hold slightly above the T axis for small $\varepsilon > 0$ values). One also finds from (4) that unstable regions do indeed emanate from the points $T = n\pi, \varepsilon = 0$; their boundaries being alternately where $\text{tr}(M) = 2$ and $\text{tr}(M) = -2$ depending whether they connect with $T = n\pi, \varepsilon = 0$ for n even or odd. These are just those regions shaded in the above figure 1 from Poincaré’s book (they are more generally, regions where ‘resonances’ produce instabilities, often called *Arnold tongues*).