An example: drawing solutions without solving the ode.[1]

A spring with friction can be modelled simply with the ode:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x = 0 \tag{1}$$

where $x \in \mathbb{R}$ is the displacement from equilibrium and $k \geq 0$ a parameter.

One can solve (1) explicitly by various methods. Here, for fun, we explain how we can also draw the form of its solutions (in the 'position-velocity' x, x' plane) without needing to compute explicit solutions, rather by applying a few general theorems on ode's.

These 'comparison methods' for obtaining (rigorously justified) sketches are more robust (ie less sensitive to changes in the differential equation) than the approach of seeking explicit solutions by quadratures.

Contents

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1 conversion to 1st order ode.

First we set $y = dx/dt$ so that in the xy plane the solutions $(x(t), x'(t))$ to (1) trace out integral curves of the line field:

$$(x + ky)dx + ydy = 0. \tag{2}$$

Remark. *The differential equations* (1) *and* (2) *are related, but not equivalent. Namely* (1) *corresonds to a* vector field *on the plane with* parametrized plane curves *as its solutions, whereas* (2) *defines a line field on the plane whose solutions are the unparametrized plane curves traced by the solutions to* (1)*.*

We ask the following usual questions:

1. What are the explicit solutions to (2)?
2. Can we reliably draw the solutions to (2)?

We will see that it is possible to answer the 2nd question (to draw the general form of the solution curves) without ever answering the first question (that is to say, without every finding the explicit formulas for the solutions).

First we note that for (2), we *can* compute the explicit solutions via the following steps:

- write in the normal form: $dy/dx = -k - x/y$, which is a homogeneous ode.
- Make the substitution $u = y/x$, to transform (2) into the separable ode $xdu/dx = -k - 1/u - u$, ie:

$$xu\ du + (u^2 + ku + 1)dx = 0 \tag{3}$$

- Separate to have the solutions upto quadratures as:

$$\int \frac{udu}{u^2 + ku + 1} + \int \frac{dx}{x} = cst.$$

  The integral with the u's can be found explicitly depending upon the roots of $u^2 + ku + 1$:

  - when $0 \le k < 2$, substitute $u + k/2 = av$ where $a = \sqrt{1 - k^2/4}$ to have $\int \frac{vdv}{1+v^2} - \frac{k}{2a} \int \frac{dv}{1+v^2} = \frac{1}{2}\log(1 + v^2) - \frac{k}{2a}\arctan(v) + c$.
  - when $k = 2$, substitute $u + 1 = v$ to have $\int \frac{(v-1)dv}{v^2} = \log|v| + \frac{1}{v} + c$,
  - when $k > 2$, substitute $u + k/2 = bv$ where $b = \sqrt{\frac{k^2}{4} - 1}$ to have $\int \frac{vdv}{v^2-1} - \frac{k}{2b}\int \frac{dv}{v^2-1}$ or (partial fractions) $\frac{1}{2}\log|v^2 - 1| - \frac{k}{4b}\log|\frac{v-1}{v+1}| + c$.

- Convert the results from the previous step back to the original variables through $u = y/x$, to have an explicit formula for the solutions in the xy-plane.

Remark. *When $k = 0$ we do not need to make any substitutions, since the original ode is separable:* $xdx + ydy = 0 \implies x^2 + y^2 = c$.

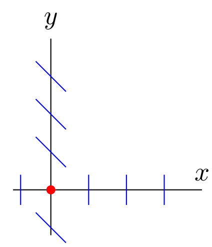
It is a good exercise in integrals to be able to find these explicit solutions (and consider carefully their dependence on initial conditions). Evidently, even for this simple looking ode, the formulas for its solutions can be somewhat complicated (and drawing the form of their graphs, even with the explicit solutions is a bit of work).

2 drawing solutions (without using the explicit solution)

Here we will instead aim to only draw the general form of the solutions to (2).

First, lets note that (2) has the origin $x = y = 0$ as a singular point. As landmarks we can see that along the axes:

- along the x-axis ($x \neq 0, y = 0$) the line field (2) is vertical (slope ∞),
- along the y-axis ($y \neq 0, x = 0$) the line field (2) has slope $-k$.



One way to decide the general form for the integral curves is to establish how the different germs of solutions emitting from the axes might (or might not) connect up.

The shapes of the integral curves will depend on the values of the parameter k (in the same way the explicit type of formula depend on k).

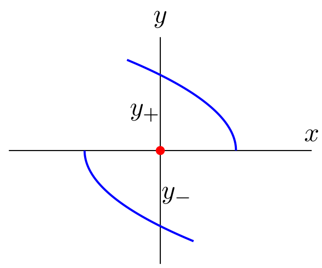
First we have the following useful facts for drawing solutions:

Claim 1. *Through each* $(x_o, y_o) \in \mathbb{R}^2 \backslash \{(0,0)\}$ *there passes a unique (connected) integral curve of* (2) *lying in* $\mathbb{R}^2 \backslash \{(0,0)\}$.

That is to say, avoiding the singular point at the origin ($x = y = 0$) the uniqueness and existence theorem applies to the line field (2). Similarly:

Claim 2. *An integral curve through* $(0, y_+)$ *with* $y_+ > 0$ *always intersects the positive* x*-axis. More precisely, following the integral curve from* $(0, y_+)$ *clockwise, its first intersection with the* x*-axis is at a point* $(x_+, 0)$ *with* $0 < x_+ < y_+$.

Similarly, an integral curve through $(0, y_-)$ *with* $y_- < 0$ *always intersects the negative* x*-axis. More precisely, following the integral curve from* $(0, y_-)$ *clockwise, its first intersections with the* x*-axis is at a point* $(x_-, 0)$ *with* $y_- < x_- < 0$.



Remark. *Claim 2 also applies to curves starting on the x-axis, eg an integral curve through* $(x_+, 0)$ *with* $x_+ > 0$ *always intersects the positive y-axis in some point* $(0, y_+)$ *with* $x_+ < y_+ < \infty$ *(followed counter-clockwise from* $(x_+, 0)$*).*

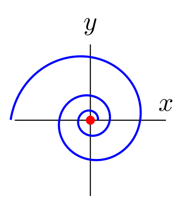
2.1 $0 < k < 2$

Here we may obtain the form of the solutions with

Claim 3. *When* $0 < k < 2$*, an integral curve through* $(x_+, 0)$ *with* $x_+ > 0$ *intersects the negative y-axis: following the integral from* $(x_+, 0)$ *clockwise, its first intersection with the y-axis is at a point* $(0, y_-)$ *with* $-x_+ < y_- < 0$.

Similarly, an integral curve through $(x_-, 0)$ *with* $x_- < 0$ *intersects the positive y-axis: following the integral from* $(x_+, 0)$ *clockwise, its first intersection with the y-axis is at a point* $(0, y_+)$ *with* $0 < y_+ < -x_-$.

Accepting these last claims 1 – 3 then, when $0 < k < 2$ we find that the solutions to (2) have the general form of spirals:

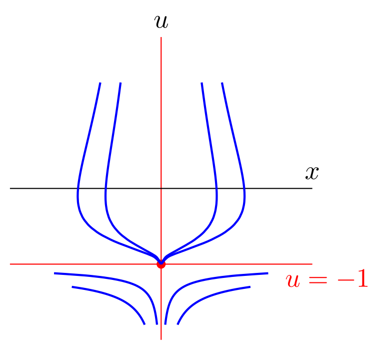


2.2 $k = 2$

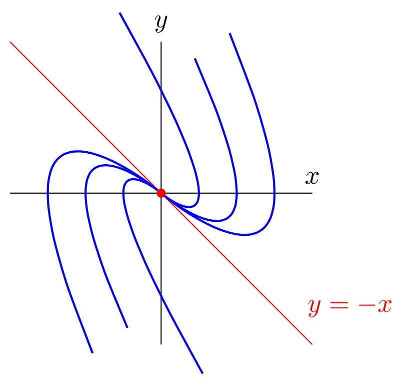
Here the ode (3) in the new coordinates $y = ux$ has the form

$$xudu/dx = -(u+1)^2.$$

There is an 'easy' solution: $u(x) \equiv -1$, or in the original coordinates the line $y = -x$ is a distinguished solution. By considering the slopes of $du/dx = -\frac{(u+1)^2}{xu}$ in the xu plane we have the general picture of these solutions (note uniqueness/existence applies also to this line field away from its singular point $x_* = 0, u_* = -1$):



Together with claims 1, 2 above we find the following picture of solutions in the original xy-plane:



2.3 $2 < k$

Here the ode (3) in the new coordinates $y = ux$ has the form

$$xudu/dx = -(u - r_1)(u - r_2)$$

for some real roots $r_1 < r_2 < 0$. There are two 'easy' solutions: $u \equiv r_1, u \equiv r_2$, or in the original coordinates the lines $y = r_j x$ (with negative slopes) are a pair of distinguished solutions. We leave for exercise to draw the pictures which follow from similar steps as the previous example: consider the slopes of $xudu/dx = -(u - r_1)(u - r_2)$ in the xu plane one can draw the general picture of these solutions (note uniqueness/existence applies also to this line field away from its singular points $x_* = 0, u_* = r_j$). And together with claims 1, 2 above one can draw the solutions in the xy-plane (similar to the previous example).

3 proofs of claims

The first claim 1 is just the existence uniqueness theorem (since away from the singular point $(0, 0)$ we have an analytic ode).

We can establish the remaining claims 2, 3 using the following comparison theorem (adapted from the notes)

Theorem 1. *For two line fields defined on some open set, $U \subset \mathbb{R}^2$, of the plane by continuous slope functions $p, P : U \to \mathbb{R}$ such that*

$$p(x, y) < P(x, y), \quad \forall (x, y) \in U.$$

Then for $y = s(x), y = S(x)$ solutions to the respective ode's with $s(x_o) = S(x_o)$ we have:

$$\begin{cases} s(x) < S(x), & x > x_o \\ s(x) > S(x), & x < x_o \end{cases}$$

for all x at which the solutions $(x, s(x)), (x, S(x)) \in U$ are defined.

So we obtain:

Proof. (of claim 2). Consider an integral curve of (2) through $(0, y_+)$ with $y_+ > 0$. We compare $dy/dx = -k - x/y = p(x, y)$ with $dy/dx = -x/y = P(x, y)$ which, since $k > 0$ has $P(x, y) > p(x, y)$ so that a solution $y = s(x)$ with $s(0) = y_+$ has $s(x) < S(x)$ for those $x > 0$ at which the solutions are defined and $S(x)$ tracing the upper half of circle centered at the origin of radius y_+. In particular $s(x)$ has its first zero somewhere before $S(x)$, that is to say somewhere in the interval $(0, y_+)$. Similarly if we start at $(0, y_-)$ with $y_- < 0$ we use that $s(x) > S(x)$ for those $x < 0$ where the solutions are defined which again forces $s(x)$ to have some zero before $S(x)$. Exercise: use the comparison theorem to show that an integral curve of (2) through $(x_+, 0)$ with $x_+ > 0$ always intersects the positive y-axis in some point $(0, y_+)$ with $x_+ < y_+ < \infty$ (followed counter-clockwise from $(x_+, 0)$). □

Proof. (of claim 3). We consider the transformed ode (3), which we write as: $dx/du = -\frac{xu}{u^2+ku+1} = p(x, u)$ (since $0 < k < 2$, the denominator never vanishes). We want to examine a solution, $x(u)$, which starts at $x(0) > 0$ and show that, as $u \to -\infty$ we have $x(u) \to 0$ and $0 < c_1 < ux(u) < c_2$ for some constants c_j (so that as we follow the integral curve $x(u), y(u) = ux(u)$ clockwise, we tend to a finite point on the negative y-axis). So, for $x > 0$ and $u < -1/k < 0$ we have:

$$p(x, u) = \frac{-xu}{u^2 + ku + 1} > \frac{-xu}{u^2} = -\frac{x}{u} = P(x, u)$$

Thus our solution $x(u)$ to $x'(u) = p(x(u), u)$ has, for all $u < -1/k$, the upper bound $x(u) < c_1/u$ for some constant $c_1 > 0$, and in particular as $u \to -\infty$ we have $ux(u) > c_1$. For the other direction, we use that, for $x > 0, u < 0$ we have

$$p(x, u) = \frac{-xu}{u^2 + ku + 1} < \frac{-xu}{u^2 + ku} = -\frac{x}{u + k} = P(x, u)$$

and thus our solution $x(u)$ to $x'(u) = p(x(u), u)$ has, for all $u < 0$, the lower bound $x(u) > \frac{c_2}{u+k}$ for some $c_2 > 0$. In particular as $u \to -\infty$ we have $ux(u) < \frac{c_2 u}{u+k} \to c_2$. Exercise: use the comparison theorem to establish the case when we start at $(x_-, 0)$ (with $x_- < 0$). □

Remark. *Another more elementary way to draw the solutions here is to use the two substitutions* $y = ux$ *as we did above (to draw the general form of the solutions away from the* y*-axis). And 'glue' these results together with those from the substitution* $x = vy$ *(to draw the general form of the solutions away from the* x*-axis).*