Center-focus problem.[1]

We consider a planar line field of the form:

$$(x + P(x,y))\ dx + (y + Q(x,y))\ dy = 0 \tag{1}$$

where P, Q are analytic with quadratic leading terms, i.e., we may write:

$$P = P_2 + P_3 + ..., \quad Q = Q_2 + Q_3 + ...$$

where $P_k(x,y), Q_k(x,y)$ are homogeneous polynomials of degree k in x, y.

When $P \equiv 0, Q \equiv 0$ vanish identically the integral curves of (1) are all circles centered at the origin. We will outline here an approach to examine whether the integral curves of (1) all remain closed when P and Q do not vanish identically.

Contents

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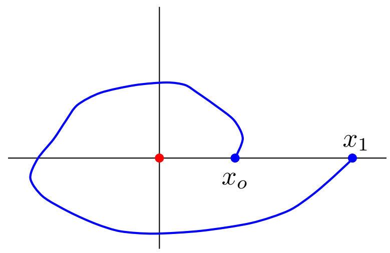
1 Poincaré return map (1st term of expansion).

To test whether the perturbed center (1) retains all closed integral curves near the origin, or is destroyed into a focus (spirals) we propose to follow an integral curve emitted say from the positive x-axis once around (counter-clockwise) and compute its next intersection point with the positive x-axis.



If this next intersection point is always the same as where we started ($x_o = x_1$) then we have still a center-like picture with all closed integral curves (otherwise, as it turns out, there is a focus like picture).

Proposition 1. *An integral curve of* (1) *with initial condition* $(x_o, 0)$ *and* $x_o > 0$ *followed counterclockwise will, for* x_o *sufficiently small, intersect again the positive* x*-axis at*

$$x_1 = x_o \left(1 + C_2 x_o^2 + ...\right)$$

where, explicitly,

$$C_2 = \int_0^{2\pi} \sin\theta P_3 - \cos\theta Q_3 + \cos(2\theta) P_2 Q_2 - \frac{1}{2}\sin(2\theta)(P_2^2 - Q_2^2)\ d\theta$$

the integrands evaluated along the unit circle: $P_k(\cos\theta, \sin\theta), Q_k(\cos\theta, \sin\theta)$.

Proof. We first transform (1) to polar coordinates, $x = r\cos\theta, y = r\sin\theta$, to have:

$$dr/d\theta = \frac{r^2\alpha(r,\theta)}{1 + r\beta(r,\theta)}$$

where $\alpha = \frac{\sin\theta P - \cos\theta Q}{r^2}$, $\beta = \frac{\cos\theta P + \sin\theta Q}{r^2}$. Since, by assumption, P, Q are analytic with quadratic leading terms we may write:

$$\alpha = \alpha_0(\theta) + r\alpha_1(\theta) + r^2\alpha_2(\theta) + ..., \quad \beta = \beta_0(\theta) + r\beta_1(\theta) + r^2\beta_2(\theta) + ...$$

where α_j, β_j are some homogeneous degree $j+3$ polynomials in $\cos\theta, \sin\theta$. To zoom in close to the origin $r = 0$, we set $r = \epsilon R$ for $\epsilon > 0$ a (small) parameter so that:

$$dR/d\theta = \epsilon \frac{R^2\alpha}{1 + \epsilon R\beta} \tag{2}$$

where $\alpha = \sum(\epsilon R)^j \alpha_j(\theta)$ and $\beta = \sum(\epsilon R)^j \beta_j(\theta)$. Since the ode is analytic (as well in the parameter) there exists a power series solution

$$R(\theta, \epsilon) = R_0(\theta) + \epsilon R_1(\theta) + \epsilon^2 R_2(\theta) + ...$$

convergent for all $\theta \in [0, 2\pi]$ and $0 < \epsilon$ sufficiently small. For initial condition $R(0, \epsilon) = R_o$ we have $R_0(0) = R_o$, $R_1(0) = 0, R_2(0) = 0, ...$ and the next intersection (Poincaré return map) of $x_o = \epsilon R_o$ is at

$$x_1 = \epsilon(R_0(2\pi) + \epsilon R_1(2\pi) + \epsilon^2 R_2(2\pi) + ...)$$

so that our task is to compute $R_j(2\pi)$. Setting $\epsilon = 0$ in (2) yields: $R_0' = 0$, so that $R_0 = cst.$, and at next order we find

$$R_1' = R_o^2 \alpha_0 \implies R_1(\theta) = R_0^2 \int_0^\theta \alpha_0(s) ds.$$

So that $R_1(2\pi) = 0$ (since $\alpha_0(s)$ is homogeneous degree 3 in $\cos s, \sin s$). Our series now reads:

$$R_o^2 \alpha_0 + \epsilon R_2' + ... = \frac{R^2 \alpha}{1 + \epsilon R \beta} = R^2 \alpha (1 - \epsilon R \beta + (\epsilon R \beta)^2 - ...)$$

differentiating both sides wrt ϵ we have:

$$R_2' + 2\epsilon R_3' + ... = (R^2 \alpha)_\epsilon (1 - \epsilon R \beta + ...) - R^2 \alpha ((\epsilon R \beta)_\epsilon - 2(\epsilon R \beta)(\epsilon R \beta)_\epsilon + ...)$$

and evaluating at $\epsilon = 0$ yields the next order term:

$$R_2'(\theta) = R_o^3 (\alpha_1(\theta) - \alpha_0(\theta) \beta_0(\theta)) + \frac{2R_1'(\theta) R_1(\theta)}{R_o}$$

$$\implies R_2(\theta) = R_o^3 \int_0^\theta (\alpha_1(s) - \alpha_0(s) \beta_0(s))\, ds + \frac{(R_1(\theta))^2}{R_o}$$

(where we have used $R_1' = R_o^2 \alpha_0$). So, over one period, we have:

$$R_2(2\pi) = R_o^3 \int_0^{2\pi} \alpha_1(s) - \alpha_0(s) \beta_0(s)\ ds$$

where $\alpha_1(s)$ is homogeneous degree 4 and $\alpha_0(s)\beta_0(s)$ is homogeneous degree 6 in $\cos s, \sin s$ (in particular they may have non-zero average). Returning to the original variables, we have the announced formula above. $\square$

Remark. *It should be clear from the last proof that it is possible* in principle *to compute as many higher order terms in the expansion*

$$x_1 = x_o(1 + C_2 x_o^2 + C_3 x_o^3 + C_4 x_o^4 + ...)$$

as necessary, but in practice *the formulas for these higher order terms will, in general, become quite complicated rather quickly.*

This last proposition we can summarize as the following test for deciding how the solutions to (1) look near the singular point. Namely, given P, Q we compute the integrals in prop. 1 to find the coefficient C_2. If $C_2 \neq 0$ then, near the origin, the return map acts as a contraction/expansion (depending if $C_2 > 0$ or $C_2 < 0$ resp.), and so we see the form of spiral integral curves emitting from the origin. If $C_2 = 0$ then the fate of the integral curves is still undecided: they *might* remain closed, but it is necessary to check also whether the remaining coefficients $C_3, C_4, ...$ of the higher order terms also vanish (in general, a task which involves infinite work!).

2 Quadratic case.

To be more explicit, we will consider the first non-trivial case when the line field (1) is not just analytic but determined by two *quadratic polynomials*:

$$P = P_2 = p_{20}x^2 + 2p_{11}xy + p_{02}y^2$$

$$Q = Q_2 = q_{20}x^2 + 2q_{11}xy + q_{02}y^2$$

for some coefficients $p_{jk}, q_{jk} \in \mathbb{R}$. In this case prop. 1 reads as:

Corollary 1. *For a quadratic line field, $P = P_2, Q = Q_2$, the return map, $x_1 = x_o(1 + C_2x_o^2 + ...)$, from prop. 1 has coefficient*

$$C_2 = \frac{\pi}{2}\left(p_{20}q_{20} - p_{02}q_{02} + q_{11}(q_{20} + q_{02}) - p_{11}(p_{20} + p_{02})\right)$$

Proof. This as an excuse to practice some trig integrals. From prop. 1 our task is to compute:

$$C_2 = \int_0^{2\pi} \cos(2\theta)P_2Q_2 + \frac{1}{2}\sin(2\theta)(Q_2^2 - P_2^2)\ d\theta$$

where

$$P_2Q_2 = p_{20}q_{20}x^4 + 2a_1x^3y + a_2x^2y^2 + 2a_3xy^3 + p_{02}q_{02}y^4$$

for $a_1 = p_{20}q_{11} + p_{11}q_{20}, a_2 = p_{20}q_{02} + 4p_{11}q_{11} + p_{02}q_{20}, a_3 = p_{11}q_{02} + p_{02}q_{11}$ and in the integrand, we evaluate at $x = \cos\theta, y = \sin\theta$. Using the double angle formulas, $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta, \sin 2\theta = 2\sin\theta\cos\theta$, the first integrand, $\cos 2\theta P_2Q_2$ consists of the terms:

$$p_{20}q_{20}\frac{(1+\cos 2\theta)^2}{4}\cos 2\theta, \quad p_{02}q_{02}\frac{(1-\cos 2\theta)^2}{4}\cos 2\theta$$

$$a_1\frac{1+\cos 2\theta}{2}\sin 2\theta\cos 2\theta, \quad a_2\frac{\sin^2 2\theta}{4}\cos 2\theta, \quad a_3\frac{1-\cos 2\theta}{2}\sin 2\theta\cos 2\theta.$$

The integrals with a_j coefficients from 0 to 2π all vanish while the remaining integrate to yield:

$$\int_0^{2\pi} \cos 2\theta P_2Q_2 d\theta = \frac{\pi}{2}(p_{20}q_{20} - p_{02}q_{02}).$$

Similarly for $\frac{1}{2}\int_0^{2\pi} \sin 2\theta(Q_2 - P_2)(Q_2 + P_2)d\theta$ we get:

$$\int_0^{2\pi} \frac{1}{2}\sin(2\theta)(Q_2^2 - P_2^2)\ d\theta = \frac{\pi}{2}\left(q_{11}(q_{20} + q_{02}) - p_{11}(p_{20} + p_{02})\right)$$

$\square$

The same remarks after prop. 1 apply here. Namely if $C_2 \neq 0$ then near the singular point, the integral curves will have the form of spirals (a focus). If $C_2 = 0$ the integral curves *might* still be closed but to be sure will require checking as well the coefficients of the higher order terms (which also must all vanish). For the present quadratic case, these remaining conditions are all some algebraic equations in the 6 variables p_{jk}, q_{jk}.

The study of the quadratic case appears to have been started by Dulac[2] and continued for example by Frommer[3] and Saharnikov[4]. Some of these older references are hard to find. From [5], [6] the quadratic case's complete answer can be summarized as:

Theorem 1. *For a quadratic line field of the form:*

$$(x + Ax^2 + (2B + \alpha)xy + Cy^2)\ dx + (y + Bx^2 + (2C + \beta)xy + Dy^2)\ dy = 0$$

then the integral curves near the origin are all closed iff one of the following three conditions is satisfied:

1. $A + C = B + D = 0$,
2. $\alpha(A+C) = \beta(B+D)$, *and* $A\alpha^3 - (3B+\alpha)\alpha^2\beta + (3C+\beta)\alpha\beta^2 - D\beta^3 = 0$
3. $\alpha + 5(B + D) = \beta + 5(A + C) = AC + BD + 2(A^2 + D^2) = 0$.

Remark. *The condition we found in Corollary 1 above is the 1st condition in item 2 above (under the notation change: $A = p_{20}, B = q_{20}, C = p_{02}, D = q_{02}, \alpha = 2(p_{11} - q_{20}), \beta = 2(q_{11} - p_{02})$).*

Remark (higher degree polynomials)**.** *One can ask for an analogue of theorem 1 for degree d polynomials, ie line fields of the form:*

$$(x + P_2 + P_3 + ... + P_d)dx + (y + Q_2 + Q_3 + ... + Q_d)dy = 0.$$

As it turns out, for $d > 2$, very little is known. About the only thing one can say in general is that for $d > 2$ there exists a finite number, $N(d)$, of conditions to determine whether the integral curves all remain closed near the origin (this is a consequence of Hilbert's basis theorem from algebraic geometry). However, already for $d = 3$ even just this number $N(3)$ of conditions is unknown!

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[2]H. Dulac. *Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre.* Gauthier-Villars (1908).

[3]M. Frommer. *Über das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung rationaler Unbestimmtheitsstellen.* Mathematische Annalen 109.1 (1934): 395-424.

[4]N.A. Saharnikov. *On Frommer's conditions for the existence of a center.* Prikl. Mat. i Meh 12.5 (1948): 669-670.

[5]W.A. Coppel. *A survey of quadratic systems.* Journal of Differential Equations 2.3 (1966): 293-304.

[6]or see §27 of C.L. Siegel, J.K. Moser. *Lectures on celestial mechanics.* Springer Science & Business Media (2012).

Remark (Hilbert's 16th problem (1900)). *While we are at it mentioning old open problems related to differential equations defined by polynomials there is the famous Hilbert's 16th problem: for a planar vector field*

$$(*) \quad dx/dt = p(x, y),\ dy/dt = q(x, y)$$

with p, q *some polynomials, say of degree* d*, in* x, y*, does there exist a number* $H(d) \in \mathbb{N}$ *such that* $(*)$ *has at most* $H(d)$ *isolated periodic orbits (limit cycles). It is known that for* p, q *of degree* d *then* $(*)$ *always has a finite number of limit cycles*[7]*, although already for* $d = 2$ *(quadratic vector fields) still nothing is known about* $H(2)$*.*

Remark (Poincaré-Dulac normal forms). *Another approach that is more adaptable to situations with more than two variables is, instead of trying to compute an expansion for a return map, to seek a series of changes of coordinates which succesively simplify the form of the line field.*

[7] This result was originally announced in:
*) H. Dulac. *Sur les cycles limites.* Bulletin de la Société Mathématique de France, 51:45–188, (1923).
However, there were gaps in Dulac's article pointed out in:
*) Yu. Il'yashenko. *Dulac's memoir "on limit cycles" and related problems of the local theory of differential equations.* Russian Mathematical Surveys, 40(6):1–49, dec (1985).
Full proofs were given in:
*) Yu. Ilyashenko. *Finiteness theorems for limit cycles*, American Mathematical Society, Providence, RI, (1991).
*) J. Ecalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Hermann, Paris, (1992).