Arnold Conjecture

For the last part of the course, we will consider developments (Floer theory) related to the Arnold conjectures. These are conjectures about the number of fixed pts. of symplectomorphisms:

ARNOLD CONJECTURE: Let (M,ω) a closed (compact and $\partial M = \emptyset$) symplectic manifold. And $\varphi: M \circlearrowleft$ a diffeo that is Hamiltonian:
($\varphi = \varphi_1$ for φ_t the flow of some Hamiltonian vector field X_t of $H: \mathbb{R}\times M \to \mathbb{R}$.)

then: $\#\,\text{fixed pts}(\varphi) \geq \min \#\{\text{crit pts. } f: M \to \mathbb{R}\}$

*) there is a weaker form of this conjecture, which asks for $\varphi: M \circlearrowleft$ a Hamiltonian diffeo all of whose fixed pts. are non-degenerate ($\det(d_x\varphi - id) \neq 0$ $x \in Fix(\varphi)$) has
$\# Fix(\varphi) \geq b_0 + \ldots + b_{2n}$ for $b_j(M)$ the Betti #s of M
(the lower bound on the # of crit pts. of a Morse function $f: M \to \mathbb{R}$).

Lets recall now some motivations for this conjecture. We have already seen, Weinstein nbhd thm, the following 'perturbative' result:

Prop: Let (M,ω) a closed symplectic manifold with $H^1(M) = 0$ (every closed 1-form on M is exact). Then any symplectomorphism $\varphi: M \circlearrowleft$; $\varphi^*\omega = \omega$ has $\# Fix(\varphi) \geq \min \#\{\text{crit pts. } f: M \to \mathbb{R}\}$
provided $\|\varphi - id\| \ll 1$ is sufficiently close to the identity map.

*) The proof of this result consists in observing that fixed points of φ correspond to intersection points of the 'graph' A_φ with the diagonal $\Delta = \{(m,m) : m \in M\} \subset M \times M$. Weinstein's neighborhood thm. for Lagrangian submanifolds then allows us to connect these fixed pts. to zeroes of a closed (and so by $H^1(M)=0$ exact) 1-form, ie to critical pt. of a function.

*) For the case of $M = S^2$, $\omega = \omega_{std}$ of the sphere with its standard area form, the conjecture is already surprising/interesting. Note that an arbitrary diffeo $S^2 \circlearrowleft$ may have exactly one fixed point. BUT if it is area preserving (and orientation preserving), then it always has at least two fixed pts. (distinct ones). [See Hofer-Zehnder]
(where we should have ≥ 3 fixed pts.)

*) For the torus, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, we can appreciate the condition that φ be Hamiltonian:

$\mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft (x,y) \mapsto (x+a, y+b) \in \mathbb{R}^2/\mathbb{Z}^2$

is a symplectic map ($dx \wedge dy$ preserving) and for general a, b without fixed pts. But it is not a Hamiltonian map

[if any translation $(x,y) \mapsto (x+a, y+b)$ was Hamiltonian, then $\iota_{(a\partial_x + b\partial_y)} dx \wedge dy$ would have to be exact on T^2].

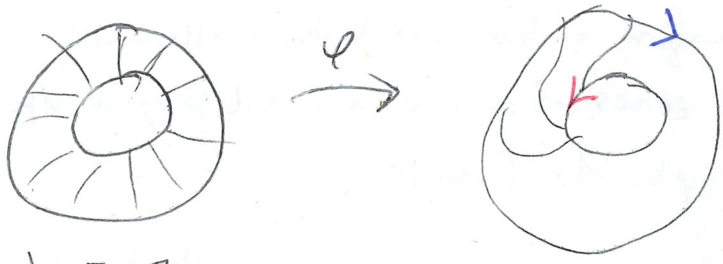
Let's consider next a certain type of fixed pts question, namely, that associated to twist maps.

TWIST MAPS

A 'twist map' is a certain map of the cylinder

$\varphi : \mathbb{R}/\mathbb{Z} \times [0,1] \circlearrowleft$

having the property that it 'turns' the boundary curves in opposite directions:



$S^1 \times [a, b]$

More precisely, we say $\varphi: \mathbb{R}/\mathbb{Z} \times [a,b] \circlearrowleft$ ~~a~~ a homeomorphism preserving the boundaries, [$\varphi(x,a) = (x', a)$, $\varphi(x,b) = (x', b)$] is a twist map of the cylinder when we have, for its lift:

$$\begin{array}{ccc} \Phi: \mathbb{R} \times [a,b] & \longrightarrow & \mathbb{R} \times [a,b] \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} \times [a,b] & \xrightarrow{\varphi} & \mathbb{R}/\mathbb{Z} \times [a,b] \end{array}$$

written as $\Phi(x,y) = (X(x,y), Y(x,y))$

with: $Y(x+1, y) = Y(x,y)$
$X(x+1, y) = 1 + X(x,y)$

we have that the twist condition:

$$(*) \qquad (x - X(x,a))(x - X(x,b)) < 0$$

holds. Then we can state:

$\boxed{\text{Theorem}}$ [Poincaré-Birkhoff] If $\varphi: \mathbb{R}/\mathbb{Z} \times [a,b] \circlearrowleft$ is an area preserving map satisfying the twist condition $(*)$. Then φ has at least two distinct fixed points.

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see Moser-Zehnder 'Notes on dynamical systems' for a proof.

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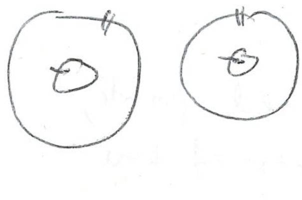
*) It's possible to show that, for smooth maps of the cylinder the twist condition (*) is equivalent [for area preserving maps] to the area preserving [symplectic] map on the torus:



induced by this twist map to be 'Hamiltonian'. So, the Arnold Conjecture, for the torus, implies the Poincaré-Birkhoff theorem. since any function $f: T^2 \to \mathbb{R}$ on the torus has ≥ 3 critical pts.

We will consider 1st a 'perturbative' version of the Poincaré-B~~irkhoff~~ theorem [analogous to our Weinstein nbhd case of Arnold conjecture]:

Prop: Let $\varphi : \mathbb{R}/\mathbb{Z} \times [a,b] \circlearrowleft$ be a diffeomorphism of the cylinder that is

1) a twist map (*)
2) Exact: $\varphi^*(y\, d\theta) = y\, d\theta + dS$ for $S(\theta, y)$ some function on the cylinder [$\mathbb{R}/\mathbb{Z} \times [a,b] \ni (\theta, y)$]
3) sufficiently close to the identity $\| \varphi - id \|_{C^1} \ll 1$.

then φ has at least two fixed points.

pf: Consider a lift $\Phi(x,y) = (X(x,y), Y(x,y))$, $\Phi : \mathbb{R} \times [a,b] \circlearrowleft$ of φ. Condition (2) on φ is then that

$$Y \cdot dX = y \cdot dx + dS \quad , \text{ for } \quad S(x+1, y) = S(x,y)$$

ie that:

$Y\cdot dX - y\cdot dx = dS$; or

since $Y\cdot dX - y\cdot dx = (Y-y)\cdot dX + (x-X)\,dy + d(y(X-x))$

we have:

$$(Y-y)dX + (x-X)dy = d(S + y(x-X)).$$

all from the exact condition (2). The fixed points, $x=X$, $y=Y$, are then exactly the critical points of the function $F(x,y) = S(x,y) + y(x - X(x,y))$.

That is, since we chose $X(x+1,y) = 1 + X(x,y)$, a 1-periodic function, i.e $F(x+1,y) = F(x,y)$ is a function on our cylinder. On the boundary, we have

$$\partial_y F(x,a) = \partial_y S(x,a) + x - X(x,a) + a(0 - X_y(x,a))$$

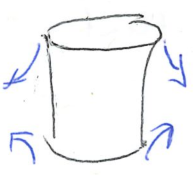
$$= (Y X_y - a X_y + x - X)\big|_{y=a} = x - X(x,a)$$

since, from $dS = YdX - ydx$, $S_y = YX_y$, and since bdrys are preserved $Y(x,a) \equiv a = y|_a$.

Likewise

$$\partial_y F(x,b) = x - X(x,b).$$

Now, by the twist condition (*), $x - X(x,a)$ and $x - X(x,b)$ have opposite signs (always). So this function F is always ~~either~~ increasing from the bdry (when $x - X(x,a) > 0$)



Being defined in a compact set, it has some interior maximum. This interior maximum is in particular a crit. pt. of F, and so a fixed pt. of φ.

But wait, $dF = 0$ implies $Y = y$ and $X = x$ only when dX, dy are independent... well this is true if φ is sufficiently close to the identity $[X = x + \varepsilon f]$. Having one fixed point we can conclude another for topological reasons. □

*) this 'perturbative' case of the Poincaré-Birkhoff thm. is due to Poincaré [and its generalization is our Weinstein nbhd prf. when $H^1(M)=0$]. This question of counting fixed pts. of maps (more precisely establishing existence of such fixed pts.) was of interest to Poincaré from his work in Celestial Mechanics.

Next lets give a non-perturbative twist map result:

Prop: Let $\varphi : \mathbb{R}/\mathbb{Z} \times [a,b] \circlearrowleft$ a diffeo of the cylinder such that

1) φ is area preserving (symplectic)

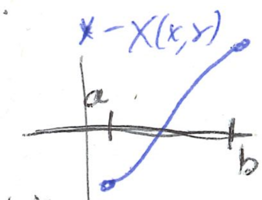
2) A Monotone twist map: for each fixed x, then

$$y \mapsto X(x,y)$$

is strictly increasing.

Then φ has at least two fixed points.

prf: Consider a lift $\Phi(x,y) = (X(x,y), Y(x,y))$ of φ.



by the twist condition:

$$(x - X(x,a))(x - X(x,b)) < 0$$

and by the monotone property, there exists then for each x a unique $y(x)$ such that $x = X(x, y(x))$.

These pairs form a graph that is mapped 'radially' under φ:

$$\Gamma = (x, y(x)) \xrightarrow{\varphi} (x, Y(x))$$

So, set $T = \{(x, y(t))\} \subset \mathbb{R}/2 \times [a,b]$,

and we have $\tilde{T} = \varphi(T) = \{(x, Y(x, y(x)))\}$

these two curves enclose the same area so must have at least two ~~fixed~~ intersection pts. These intersection pts correspond to fixed pts of φ:

$(T \cap \tilde{T} \ni (x,y) \Leftrightarrow X(x,y) = x,\ Y(x,y) = y)$ $\square$