(Hamiltonian) Floer theory

We will outline / sketch the Floer theory approach to the following case of Arnold's conjecture:

Arnold Conj.: Let (M,ω) a closed ($\partial M = \emptyset$) compact sympl. mfld. with $\pi_2(M)=0$, and $\varphi : M \circlearrowleft$ a non-degenerate Hamiltonian map: generated by $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ as the time 1-map $\varphi = \varphi_1$ of the Hamiltonian flow φ_t of $H(t,x)$, and so that every fixed pt. of φ is non-degenerate:

$$\varphi(x) = x \Rightarrow \det(d_x\varphi - id) \neq 0 \qquad (d_x\varphi : T_xM \circlearrowleft).$$

Then: $\#$ fixed pts. of $\varphi \geq b_0 + b_1 + \dots + b_{2n}$

for $b_i(M)$ the Betti #s of M.

* Note: fixed pts. of $\varphi = \varphi_1$ correspond to 1-periodic orbits of X_H (the Hamiltonian v.f. of H).

* $\pi_2(M)=0$ means any $f : S^2 \to M$ extends to $F : B^3 \to M$ with $F|_{\partial B^3} = f$. It can be replaced by a weaker conditions, such as $\int_{S^2} f^*\omega = 0 \quad \forall f : S^2 \to M$.

* the assumption $\pi_2(M)=0$ allows us to characterize these 1-periodic orbits (more precisely the contractible ones) as critical points of our Loop action functional. So, we attempt to count critical pts. of this Loop action functional using a Morse theory approach.

* Note that since M is compact and $\partial M = \emptyset$, ω is not exact (Stokes)

The loop action functional is defined as follows. ②

Let $x : \mathbb{R}/\mathbb{Z} \to M$, $x(t+1) = x(t)$ be a <u>contractible</u> loop in M, so that there exists an extension $\tilde{x} : D^2 \to M$ with $\tilde{x}|_{\partial D^2} = x$

then we take:

$$(*) \quad A_H(x) := \int_{D^2} \tilde{x}^* \omega - \int_0^1 H(t, x(t))\, dt.$$

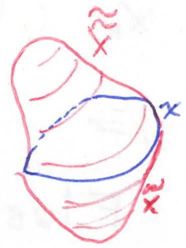
Note that if $\omega = d\lambda$ was exact then we would have

$$\int_{D^2} \tilde{x}^* \omega = \int_{D^2} d\, \tilde{x}^* \lambda = \int_{\partial D^2} \tilde{x}^* \lambda = \int_x \lambda$$

as our 'usual' loop action functional for exact symplectic mflds.

The condition $\pi_2(M) = 0$ gives that $(*)$ is well-defined.

If $\tilde{\tilde{x}} : D^2 \to M$ has $\tilde{\tilde{x}}|_{\partial D^2} = x$ is some other extension, then by gluing $\tilde{x}, \tilde{\tilde{x}}$ along ∂D^2 to a map $f : S^2 \to M$ we have



$$\int_{D^2} \tilde{x}^* \omega - \int_{D^2} \tilde{\tilde{x}}^* \omega = \int_{S^2} f^* \omega = 0$$

Since $\pi_2(M) = 0$ means $f : S^2 \to M$ extends to some $F : B^3 \to M$ with $F|_{\partial B^3} = f$ and so by $d\omega = 0$:

$$\int_{\partial B^3 = S^2} f^* \omega = \int_{B^3} F^* d\omega = 0.$$

So, the condition $\pi_2(M) = 0$ has given us a well-defined contractible loop action functional:

$$A_H : C^\infty_{\mathrm{contr}}(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$$

$$x \mapsto \int_{D^2} \tilde{x}^* \omega - \int_0^1 H(t, x(t))\, dt.$$

Exercise: Let $\varepsilon \mapsto x_\varepsilon \in C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M)$ a smooth variation of 1-periodic contractible loops through $x_0 = x$.

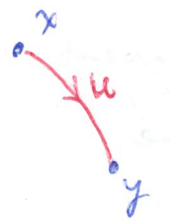
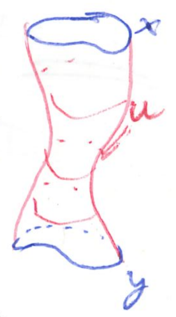
For $\xi(t) = \xi(t+1) = \frac{d}{d\varepsilon}\Big|_0 x_\varepsilon(t) \in T_{x(t)}M$ the 1-periodic vector field along $x(t)$ generated by this variation x_ε, verify that: [with $\iota_{X_H}\omega = -dH$]

$$\frac{d}{d\varepsilon}\Big|_0 A_H(x_\varepsilon) = \int_0^1 \omega(X_H - \dot{x}, \xi)\, dt =: d_x A_H(\xi).$$

So, from this last exercise, we have

$$\text{crit pts of } A_H \longleftrightarrow \text{contractible 1-periodic orbits of } X_H$$
$$\longrightarrow \text{fixed pts. of } \varphi = \varphi_1 \ (\varphi_t \text{ flow of } X_H).$$

The Morse $\longleftrightarrow$ Floer analogy then consists in:

	MORSE	FLOER
(0)	$f: M \to \mathbb{R}$	$A_H: C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$
(1)	GRADING of crit pts of f by Morse index, $\mathrm{ind}(x)$, $MC_k := (\mathbb{Z}/2\mathbb{Z})^{c_k(f)}$ for $c_k(f) = \#\{x \in \mathrm{Crit}(f) : \mathrm{ind}(x) = k\}$	GRADING of crit. pts. of A_H by CONLEY-ZEHNDER index, $\mu_{CZ}(x)$, $FC_k := (\mathbb{Z}/2\mathbb{Z})^{p_k(H)}$ for $p_k(H) := \#\{x \in \mathrm{Crit}(A_H) : \mu_{CZ}(x) = k\}$
(2)	Boundary operators: $\cdots \xrightarrow{\partial_{k+1}} MC_k \xrightarrow{\partial_k} MC_{k-1} \to \cdots$ are defined using a 'generic' choice of Riem. metric g (Morse-Smale) on M and 'counting' gradient flow lines $u' \overset{(*)}{=} -\nabla^g f(u)$ between crit pts: $\partial x = \sum_{\mathrm{ind}(y) = \mathrm{ind}(x) - 1} \nu(x,y)\, y$ 	$\cdots \xrightarrow{\partial_{k+1}} FC_k \xrightarrow{\partial_k} FC_{k-1} \to \cdots$ are defined using a 'generic' choice of $J \in \mathcal{J}(M, \omega)$, which induces an L^2 inner product on $C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M)$, and we 'count' solutions of $u' \overset{(*)}{=} -\nabla^J A_H(u)$ between crit. pts: $\partial x = \sum_{\mu_{CZ}(y) = \mu_{CZ}(x) - 1} \nu(x,y)\, y$ 

$$\nu(x,y) := \#\{\text{gradient flowlines of } (*) \text{ from } x \to y\} \pmod 2$$

Provided one has defined all these items (0)–(2), one then shows:

(3) $\partial^2 = 0$

so that we have a complex and, the homology:

$$MH_k := \ker\partial_k / \operatorname{im}\partial_{k-1} \qquad FH_k := \ker\partial_k / \operatorname{im}\partial_{k-1}$$

which may a-priori depend upon all of the choices made so far
(f, g for Morse and H, J, ω for Floer).

(4) Finally one needs to show an <u>invariance</u> upon these choices:
for Morse that $MH_k(f,g,M) \cong H_k(M)$, and for Floer,
it is done by 1st showing
$$FH_k(M,\omega,H,J) \cong FH_k(M,\omega,H) \equiv FH_k(M,\omega)$$
does not depend upon the choice of 'generic' J or non-degenerate H,
Finally, one computes by taking a <u>certain</u> choice of H, J
that $FH_k(M,\omega) \cong H_{k+n}(M)$ $\quad [n = \frac{1}{2}\dim M]$
This then establishes our 'non-degenerate' case of the Arnold conjecture, since the ranks total of FC_k's are $\geq$ rank total FH_k's
which reads: $\#\,\text{fixed pts}\,\varphi = \sum p_k(H) \geq b_0 + \dots + b_{2n}$

$(b_j = \operatorname{rank} H_k(M))$.

Here we are ignoring orientations and working mod 2 w/ $\mathbb{Z}/2\mathbb{Z}$.
Its possible to introduce signs via orientations to work over $\mathbb{Z}$.

In the remainder, we comment on what is involved for steps (0)–(4) above.

(ii) In the Morse case, the non-degeneracy (Morse condition) of $f: M \to \mathbb{R}$ ($d_x^2 f$ invertible each $x \in \mathrm{crit}(f)$) yields that critical pts. of f are isolated, and by compactness of M, there are a finite #, so that $C_k(f)$ are all finite.

In the Floer case, it is the same, since by non-degeneracy assumption on φ, its fixed pts. are all isolated, and so there are a finite # by cptness of M and so $P_H(H)$ are all finite, and the FC_k's are well-defined.

As for why in the Floer case we need C-Z index and not some 'naive' extension of Morse index of A_H, let us recall that the Morse index of f at crit pt. x is

$$ind_f(x) = \max\{\dim V : V \subset T_x M,\ d_x^2 f(v,v) < 0\ \forall v \in V \setminus 0\}$$

For our contractible loop action functional, such an index is generally not finite, and so useless for determining a GRADING:

$\boxed{\text{PROP:}}$ Let $x \in C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M)$ be a critical point of A_H. Then $\max\{\dim V : V \subset T_x C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M)$ with $d_x^2 A_H|_V$ negative definite$\} = \infty$.

Consider for simplicity $x \in \mathrm{Crit}(A_H)$ contained in a Darboux chart with $\omega = d\lambda$ in this chart.

$$x(t) = x(t+1) \in \mathbb{R}^{2n} = \mathbb{C}^n$$

$$[\ \lambda_z(\dot z) = \frac{z \cdot i\dot z}{2} = \frac{p\dot q - q\dot p}{2}\ ]$$

Then, for $\xi(t+1) = \xi(t) \in \mathbb{C}^n$, considered as a 1-periodic vector field along $x(t)$ $[\ T_x C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M) \ni \xi\ ]$ we have

$$A_H(x + \varepsilon\xi) = A_H(x) + \varepsilon\, d_x A_H(\xi) + \frac{\varepsilon^2}{2}\int_0^1 \xi \cdot i\dot\xi - d_x^2 H(\xi,\xi)\,dt + O(\varepsilon^3)$$

$$\Longrightarrow\ d_x^2 A_H(\xi,\xi) = \int_0^1 \xi \cdot i\dot\xi - d_x^2 H(\xi,\xi)\,dt.$$

Now, we take Fourier series:

$$\xi(t) = \sum_{k \in \mathbb{Z}} \xi_k e^{-2\pi i k t}, \quad \text{so that}$$

$$d_x^2 A_H(\xi,\xi) = 2\pi \sum_{k \in \mathbb{Z}} k |\xi_k|^2 - \int_0^1 d_x^2 H(\xi,\xi)\, dt$$

Since the periodic orbit $x(t)$ at which we evaluate the quadratic form $d_x^2 H$ is fixed, we have uniform bounds:

$$|d_x^2 H(\xi,\xi)| \leq C|\xi|^2 \quad \text{some constant } C > 0,$$

and so $d_x^2 A_H(\xi,\xi) \leq \sum_{k \in \mathbb{Z}} (2\pi k + C)|\xi_k|^2$ is negative definite on the ∞-dimensional space $\xi_k = 0$ $k \geq -C/2\pi$ (and $\xi_k \neq$ some $k < -C/2\pi$). $\square$

So, for step (1) to define a Grading, we need some other type of index (the CZ-index) which we will come back to.

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(2) Pick some ω-compatible $J \in \mathcal{J}(M,\omega)$, so we have a Riem. metric $g_J(u,v) = \omega(Ju,v)$ $[\omega(u,v) = g_J(u,Jv)]$ on M. The induced metric on $C^\infty_{contr}(\mathbb{R}/\mathbb{Z}, M)$ is then:

$$\langle \xi_1, \xi_2 \rangle_x := \int_0^1 g_J(\xi_1,\xi_2)\, dt \qquad [\xi_k(t) \in T_{x(t)}M].$$

For the gradient of A_H wrt this metric we have (from exercise pg. 3):

$$d_x A_H(\xi) = \int_0^1 \omega(X_H - \dot{x}, \xi)\, dt = \int_0^1 g_J(X_H - \dot{x}, J\xi)\, dt$$

$$= \int_0^1 g_J(J(\dot{x} - X_H), \xi)\, dt = \langle J(\dot{x} - X_H), \xi \rangle_x$$

so that: $$\nabla^J A_H = J(\dot{x} - X_H) = J\dot{x} - \nabla_x H$$

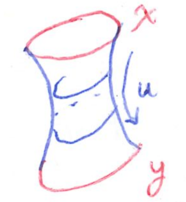
[since $\nabla H = JX_H$ for ∇ the g_J-gradient of H].

The gradient equation, $u' = -\nabla^J A_H$, is then:

$$(*) \quad \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = \nabla_u H \qquad [\text{Floer eqn}]$$

for $u(s,\cdot): \mathbb{R}/\mathbb{Z} \to M$ a curve of loops in M $\left(\begin{array}{l} u: \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M \\ (s,t) \mapsto u(s,t) \end{array} \right)$.

Solutions of the Floer equation (*) are our analogues of gradient lines in Morse theory. In particular, to define our boundary operator ∂ of the Floer complex FC_k, we need to consider the spaces:



$$\tilde{\gamma}(x,y) = \left\{ u \text{ solving } (*) : \begin{array}{l} \lim_{s\to\infty} u(s,t) = y(t) \\ \lim_{s\to -\infty} u(s,t) = x(t) \end{array} \right\}$$

for $x, y \in \mathrm{Crit}(A_H)$. This is the space of <u>parametrized</u> gradient lines from x to y. Note that if $u(s,t) \in \tilde{\gamma}(x,y)$ then for any $c \in \mathbb{R}$ so is $u(s+c,t)$. The quotient space:

$$\gamma(x,y) := \tilde{\gamma}(x,y)/_{\mathbb{R}} = \tilde{\gamma}(x,y)/\{u(s,t) \sim u(s+c,t)\}$$

are (for $\mu_{CZ}(y) = \mu_{CZ}(x) - 1$) the spaces we would like to count to define $\nu(x,y)$ and ~~our~~ our bdy operator ∂.

In Morse theory, for g s.t. ∇f is Morse-Smale, these analogous spaces are $\tilde{\gamma}(x,y) = W^u(x) \cap W^s(y)$ and have dimension $\mathrm{ind}(x) - \mathrm{ind}(y)$, so that $\dim \gamma(x,y) = \mathrm{ind}(x) - \mathrm{ind}(y) - 1$.

A key property of the Conley-Zehnder indices is:

Prop: For a 'generic' choice of $J \in \mathcal{J}(M,\omega)$, $\gamma(x,y)$ is a smooth mfld of dimension: $\dim \gamma(x,y) = \mu_{CZ}(x) - \mu_{CZ}(y) - 1$.

In particular, when $\mu_{CZ}(y) = \mu_{CZ}(x) - 1$, then $\dim \gamma(x,y) = 0$ is a discrete set. Pending a compactness property then we have $\nu(x,y) = \# \gamma(x,y) \bmod 2$ to have our ∂'s well-defined.

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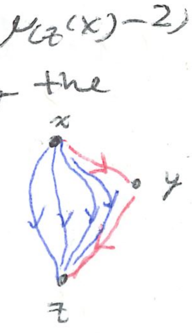
(3) That $\partial^2 = 0$ is related to showing compactness of $\gamma(x,y)$'s. Namely, one shows that for $\mu_{CZ}(z) = \mu_{CZ}(x) - 2$, the 1-dimensional space $\gamma(x,z)$ has as boundary the concatenated gradient lines $x \to y \to z$, $\mu(y) = \mu(x) - 1$.



(4) Accepting the invariance of FH_k on J's and H's, let's see how one can take a particular choice in order to compute $FH_* \approx H_{*+}(M)$ [up to shifts].

For our Hamiltonian, we take an autonomous

$$H_0 : M \to \mathbb{R}$$

which is a Morse function on M, and 'sufficiently C^2-small' so that the time 1-map of its flow $\varphi = \varphi_1$ is non-degenerate:

$\boxed{\text{Prop:}}$ If $H_0 : M \to \mathbb{R}$ is a Morse function with $|H_0|_{C^2} \ll 1$ suff. small, the $\varphi = \varphi_1$, for φ_t the flow of X_{H_0}, has as fixed pts only the non-degenerate critical pts. of H_0.

pf: By $\|H_0\|_{C^2} \ll 1$ we mean that we can take some (Darboux) atlas on M with $|H_0|_{C^2} \ll 1$ in each chart. So, we will show the above claim for $H_0 : \mathbb{R}^{2n} \to \mathbb{R}$, $|H_0|_{C^2} \ll 1$.

Suppose, for contradiction, $x(t+1) = x(t)$ is a 1-periodic orbit of X_{H_0}: $\dot{x} = -i \nabla H_0(x)$ and consider its Fourier series:

$$x(t) = \sum x_k e^{-2\pi i k t}, \quad \dot{x}(t) = \sum -2\pi i k x_k e^{-2\pi i k t} = -i \nabla H_0(x)$$

$$\ddot{x}(t) \overset{(*)}{=} \sum 4\pi^2 k^2 x_k e^{-2\pi i k t} \overset{(*)}{=} -i \nabla_x^2 H_0(\dot{x}).$$

Then $\|\ddot{x}\|_{L^2}^2 \overset{(*)}{=} 4\pi^2 \sum k^2 |2\pi i k x_k|^2 \geq \|\dot{x}\|_{L^2}^2$

and $\|\ddot{x}\|_{L^2}^2 \overset{(*)}{\leq} \varepsilon \|\dot{x}\|^2$ for $\varepsilon \ll 1$ (since $|\nabla_x^2 H_0| \ll 1$), so $x(t) \equiv x_0$ is constant

and so $0 = -i \nabla H_0(x_0)$, is a crit pt. of H_0. $\square$

So, for this choice of H_0, the elements of the complexes MC_* and FC_* are the same: the crit. pts. of H_0.

As for the gradings:

Prop: For $x_0 \in Crit(H_0)$ a non-degenerate critical point of $H_0 : M \to \mathbb{R}$ with $|H_0|_{C^2} << 1$ as above, then when viewed as a 1-periodic orbit of X_{H_0}, we have:

$$\mu_{CZ}(x_0) + n = ind(x_0) \qquad \left[n = \tfrac{1}{2} \dim M \right]$$

So, the Morse and Floer complexes are for this choice the same up to an index shift:

$$MC_{k+n}(H_0) = FC_k(H_0)$$

Lets consider this index relation in the case $n=1$ ($\dim M = 2$).

The C-Z index of a path $[0,1] \ni t \mapsto A(t) \in Sp_2(\mathbb{R}) = SL_2(\mathbb{R})$, $A(0)=I$, and with $\det(A(1) - I) \neq 0$ is defined by ~~completing~~ extending $A(t)$ to a path which induces a loop in the Lagr. Grassmannian $\Lambda_1 = \mathbb{R}P^1$, and computing the Maslov index ($\tfrac{1}{2}$-turns) of this induced loop.

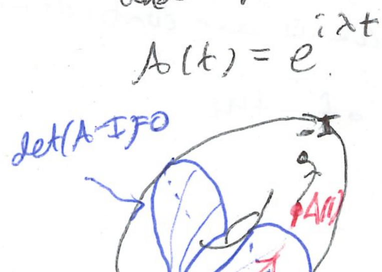
Namely, if $A(1) \in Sp^+ = \{\det(A-I) > 0\}$ we extend to connect to $-I$. while if $A(1) \in Sp^- = \{\det(A-I) < 0\}$, we extend to connect to I (or $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$).

Given a 1-periodic orbit, $\dot{x} = X_H$, we induce this path $A(t)$ as the linearized flow: consider the linearized system

$$\dot{\xi} = -i \nabla^2_x H \cdot \xi \quad \text{of} \quad \dot{x} = -i \nabla_x H, \text{ w/ fundamental matrix}$$

solution $A(t)$ solving $\dot{A} = -i \nabla^2_{x(t)} H \cdot A$, $A(0) = Id$.

Now, for a maximum x_0 of H_0, we have $ind_{Morse}(x_0) = 2$, and for its C-Z index; $\dot{\xi} = i\lambda \xi$ some $0 < \lambda << 1$ so that $A(t) = e^{i\lambda t}$ is rotating by λt:



closing it to end at $-I$, we have a rotation by π, so one $\tfrac{1}{2}$-turn $\equiv$ 1 turn in Λ_1 and $\mu_{CZ}(x_0) = 1$ $(= 2 - 1)$.

Likewise, if x_0 is a minimum of H_0, so $\text{ind}_{Morse}(x_0) = 0$,

then we have a path $A(t) = e^{-i\lambda t}$, $0 < \lambda t < 1$, which extends to $-I$

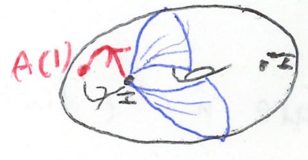
~~done~~ by completing one ½ twist in the opposite sense in Λ_1, so that

$\mu_{CZ}(x_0) = -1$ $(= 0 - 1)$.

And, if x_0 is a saddle pt. of H_0, so $\text{ind}_{Morse}(x_0) = 1$, we

write $H_0 = xy + \dots$ and have linearized system

$\dot{x} = -\lambda x$
$\dot{y} = \lambda y$
with the path $A(t) = \begin{pmatrix} e^{-\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$, which closes to a path ~~that is~~ contractible, so $\mu_{CZ}(x_0) = 0 = 1 - 1$ ✓.



Finally, we claim the bounding operators in this case are the same.

For this, we observe that for this $|H_0| \ll 1$ and some 'generic'

choice of $J \in \mathcal{J}(M, \omega)$ then the solutions $u(s,t)$ of the Floer eq'n

from $x \to y$ are time independent: $u(s)$, and

so are just gradient lines of H_0. In particular, the

Morse and Floer boundary operators in this case are just counting

the same count and we get that $MH_{k+n}(H_0) = H_{k+n}(M) = FH_k(H_0)$.

To see this time independence, we will consider J so that

the dimension count: $\dim \mathcal{M}(x,y) = \mu_{CZ}(x) - \mu_{CZ}(y) - 1$

holds. In particular, for $\text{index}(y) = \text{index}(x) - 1$ then this is a

discrete 0-dimensional space. But, if $u(s,t)$ depended on t, then

since $H_0 : M \to \mathbb{R}$ is autonomous, we would have for any $a, b \in \mathbb{R}$

also the solution $u(s+a, t+b) \in \widetilde{\mathcal{M}}(x,y)$, and then

$\dim \widetilde{\mathcal{M}}(x,y) \geq 2$ and so too $0 = \dim \mathcal{M}(x,y) \geq 1$ which is our contradiction

of dimension formula, and so indeed the solutions of the

Floer eq'n in this case are all independent of t.

Conley-Zehnder Index

Now we will re-visit the property for counting dimensions, and give a definition of C-Z index. Recall we are interested in this index because we claim for "generic" J's that:

$$\dim \mathcal{M}(x,y) = \mu_{CZ}(x) - \mu_{CZ}(y) - 1 .$$

To set up this 'count', let us fix the two orbits $x, y \in \mathrm{Crit}(\mathcal{A}_H)$, and write

$$\mathcal{Z} = C^\infty(\mathbb{R}\times\mathbb{R}/\mathbb{Z}, M)_{x,y}$$

for the parametrized 'cylinders' $(s,t) \mapsto u(s,t) \in M$ such that

$$\lim_{s\to\infty} u(s,t) = y(t) \quad \text{and} \quad \lim_{s\to-\infty} u(s,t) = x(t).$$

The Floer eq'n we think of as a vector field on $\mathcal{Z}$:

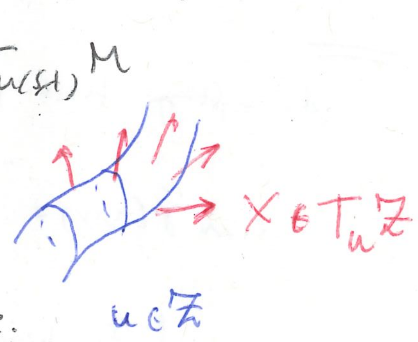
$$\mathcal{F} : \mathcal{Z} \longrightarrow T\mathcal{Z}$$

$$u \longmapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} - \nabla_u H$$

where we consider the fibers $T\mathcal{Z} \to \mathcal{Z}$ over $u \in \mathcal{Z}$ as the tangent spaces:

$$T_u\mathcal{Z} := \left\{ X : \mathbb{R}\times\mathbb{R}/\mathbb{Z} \xrightarrow{C^\infty} u^*(TM) \;\; X(s,t) \in T_{u(s,t)}M \right\}$$

as vector fields along the cylinders.

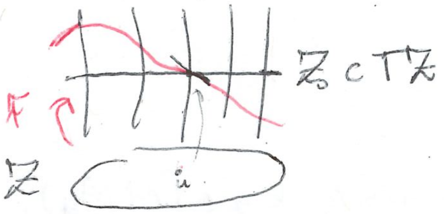


The solutions to the Floer eq'n we are interested in 'counting' are the zeros of $\mathcal{F}$:

$$\{u : \mathcal{F}(u) = 0\} = \mathcal{M}(x,y).$$

The approach is to use implicit function theorem.

This level set $\mathcal{F}^{-1}(0)$ is the intersection of $\mathrm{im}(\mathcal{F})$ with the zero section, so our implicit func thm. consists in trying to show these intersection points are transverse.



That is one wants to show, for $\mathcal{F}(u)=0$

*) $d_u\mathcal{F} : T_u\mathcal{Z} \to T_u\mathcal{Z}$ is onto

*) and compute $\dim \ker d_u\mathcal{F} = \dim \tilde{\mathcal{Y}}(x,y)$

Prop: For 'generic' $J \in \mathcal{J}(M,\omega)$, $d_u\mathcal{F}$ is a Fredholm operator, and it is surjective.

*) The domain of $\mathcal{F}$ is extended to a sobolev space of mappings so that we can work with certain Hilbert spaces. $W^{1,2}(\mathbb{R}\times\mathbb{R}/\mathbb{Z}, M)$

*) Recall that a linear operator $A: H_1 \to H_2$ (between Hilbert spaces H_1, H_2) is called Fredholm if: 1) it is bounded 2) its image is closed 3) $\dim \ker A < \infty$ 4) $\dim \mathrm{coker} A = \dim(H_2/\mathrm{Im}(A)) < \infty$

The *index* of a Fredholm operator A is

$$\mathrm{ind}_{Fred}(A) = \dim \ker A - \dim \mathrm{coker}(A).$$

Accepting this proposition, our sought dimension is then

$$\dim \tilde{\mathcal{Y}}(x,y) = \dim \ker d_u\mathcal{F} = \mathrm{ind}_{Fred}(d_u\mathcal{F})$$

since $d_u\mathcal{F}$ being onto, we have $\dim \mathrm{coker}\, d_u\mathcal{F} = 0$.

So, we want to find the index of some Fredholm operator

For ~~the~~ index of Fredh. operator lets recall:

*) $\text{ind}_{\text{Fred}}(A + K) = \text{ind}_{\text{Fred}}(A)$ for A Fredholm and K compact operators

*) $\text{ind}_{\text{Fred}}(A + D) = \text{ind}_{\text{Fred}}(A)$ with $\|D\| \ll 1$ suff small.

Computing this index of $d_u\mathcal{F}$ at $u: \mathcal{F}(u) = 0$ consists in using these "perturbative stability" of Fredholm index to replace $d_u\mathcal{F}$ with a simpler Fredholm operator, L, which at the end of the day one can actually compute.

handy $\text{ind}_{\text{Fred}}(d_u\mathcal{F}) = \text{ind}_{\text{Fred}}(L)$, and

To start to set up the problem; one 1st computes:

Prop: Choose a symplectic trivialization:

$$u^*(TM) \simeq (\mathbb{R}\times S^1)\times \mathbb{R}^{2n}$$

so that $T_u\mathcal{Z} = C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \ni X$, with $\frac{\partial X}{\partial s} \xrightarrow[s\to\pm\infty]{} 0$. The linearized Floer equation is given in these coordinates by:

$$(*) \quad d_u\mathcal{F}(X) = \frac{\partial X}{\partial s} + i\frac{\partial X}{\partial t} - \nabla_u^2 H\cdot X + P(u)\cdot X$$

and where $P(s,t) = P(u(s,t)) \longrightarrow 0$ as $s \to \pm\infty$.

Now, using $P(s,t) \xrightarrow{s\to\pm\infty} 0$, and that $u(s,t) \xrightarrow{s\to\infty} y(t)$, $u(s,t) \xrightarrow[s\to-\infty]{} x(t)$,

one can show that $\text{ind}_{\text{Fred}}(d_u\mathcal{F}) = \text{ind}_{\text{Fred}}(L)$ where

$$L(X) = \begin{cases} \frac{\partial X}{\partial s} + L^+(X) & ; \ s \gg 0 \\ \frac{\partial X}{\partial s} + L^-(X) & ; \ s \ll 0 \end{cases}$$

for $L^{\pm}(X) = i\dfrac{\partial X}{\partial t} - S^{\pm}(X)$

and $S^{+} = \nabla_y^2 H$; $S^{-} = \nabla_x^2 H$ ($2n\times 2n$ symmetric t-dependent matrices)

For $|s| < s_0$ these two 'asymptotic' operators are 'interpolated' by the Fredholm operator L.

Now, the operators of the form $L^{\pm}$, are closely related to the Conley-Zehnder index, in particular the eigenvalues of these operators.
To see how these eigenvalues may play a role in computing $\mathrm{ind}(L)$; eg $\ker(L)$, Note 1st that since $S^{\pm}(t)$ are s-independent, we may view $L^{\pm}$ as operators on $C^{\infty}(S^1, \mathbb{R}^{2n})$:

$$\xi(t) \xrightarrow{L^{\pm}} i\dot{\xi} - S^{\pm}\cdot\xi .$$

Suppose $\eta^{\pm}$ are eigenvalues of $L^{\pm}$ with eigenvectors $\xi^{\pm}(t)$.
Then we have (separation of variables) a solution

$$X^{\pm}(s,t) = e^{-\eta^{\pm} s}\, \xi^{\pm}(t)$$

to $\dfrac{\partial X}{\partial s} + L^{\pm}(X) = 0$.

That is, if we take an eigenvalues $\eta^{+} > 0$, $\eta^{-} < 0$ then we have two functions $X^{\pm}(s,t)$ with

$L X^{\pm}(s,t) = 0$ for $s \gg 0$ (X^{+}) or $s \ll 0$ (X^{-}).
and which have each the proper decay $\partial_s X^{\pm} \to 0$, $s \to \pm\infty$ at the ends. Analyzing which such eigenvalues $\eta^{\pm}$ correspond to actual elements of $\ker L$ involves more detailed study of this 'interpolating' region.

But, at least, we can see the eigenvalues of these operators L^I should play a role.

So, now, we will focus on operators of the following form:

$$L_S : C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \circlearrowleft \qquad \xi(t) \mapsto i\dot{\xi}(t) - S(t)\cdot\xi(t)$$

where $t \mapsto S(t) \in \mathrm{Sym}(2n\times 2n)$ is a (smooth) path of $2n\times 2n$ symm matrices.

* we can also consider L_S on $W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \to L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$

$\boxed{\text{Prop:}}$ An operator $L_S = i\frac{d}{dt} - S : W^{1,2} \to L^2$ is Fredholm, and has $\mathrm{ind}_{Fred}(L_S) = 0$.

pf: Since $|S\cdot\xi| \le C|\xi|$ for some constant C, we have that $\xi \mapsto S\cdot\xi$ is compact operator ($W^{1,2} \subset L^2$ is rel. cpt.).

So $L_S = L_0 + \text{Cpt.}$ and if we show $L_0 = i\frac{d}{dt}$ is Fredholm, then so is L_S. So lets check L_0 is Fredholm:

1) it is bdd ✓ $\|L_0\xi\|_{L^2} \le \text{const} \|\xi\|_{W^{1,2}}$

2) its image is $\sum_{k\neq 0} x_k e^{2\pi ikt} \in L^2$ $(\sum |x_k|^2 < \infty)$ which is closed, and so $\dim\operatorname{coker} L_0 = 2n$

3) its kernel is $x(t) \equiv x_0 \in \mathbb{R}^{2n}$ the constant loops, and has $\dim\ker L_0 = 2n$,

So, in particular $\mathrm{ind}(L_0) = \mathrm{ind}(L_S) = 0$. $\square$

*) We also can check that L_S is symmetric:

$$\langle L_S\xi_1, \xi_2\rangle_{L^2} = \langle \xi_1, L_S\xi_2\rangle_{L^2}.$$

These operators L_S are related to periodic orbits, and paths of symplectic matrices as follows.

1) Let $t \mapsto A(t) \in Sp_{2n}(\mathbb{R})$ a path of symplectic matrices, say with $A(0) = Id$.

Then we have $B(t) = \dot{A}(t) A^{-1}(t) \in sp_{2n}(\mathbb{R})$.

In particular, we can write
$B(t) = -JS(t)$ for some $2n \times 2n$ symmetric matrices $S(t)$.

Then we can consider L_S for these symmetric matrices.

The symplectic matrices are recovered from $S(t)$ as the solution

$$\dot{A} = JB(t)A \quad \text{w/} \quad A(0) = Id.$$

2) In particular for $x(t+1) = x(t)$ a periodic orbit of X_H, we have the path of symplectic matrices

$d_{x(0)}\varphi_t : T_{x(0)}M \to T_{x(t)}M$ that we can write as $A(t) \in Sp_{2n}(\mathbb{R})$

by choosing a (symplectic) trivialization of $x^*(TM) \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n}$

(always possible if eg x is contractible).

In case we are in $\mathbb{R}^{2n}$ then $A(t) = d_{x(0)}\varphi_t : \mathbb{R}^{2n} \circlearrowleft$

satisfies the linearized eq's along the orbit:

$$\dot{\xi} = -i \nabla^2_{x(t)} H \cdot \xi \qquad (\text{and } S(t) = \nabla^2_{x(t)} H).$$

or $\quad i\dot{\xi} + S \cdot \xi = 0$.

Note: the periodic orbit is non-degen when $|A(1) - id| \neq 0$.

3) $\ker L_S \cong \ker(A(1) - id)$.

4) the spectrum of L_S are all actual eigenvalues λ;
$L_S - \lambda I$ is still Fred, and index 0, if $\ker = 0$ then $\operatorname{coker} = 0$ then $\Rightarrow$ invertible.

5) The spectrum of L_S consists of discrete countable, real eigenvalues

6) They can be indexed uniquely by:

Prop: Given L_S, there is a unique way to order its spectrum:

$$\dots \eta_{-1}(S) \leq \eta_0(S) \leq \eta_1(S) \leq \dots$$

such that

1) $\eta_k(S)$ depends continuously on S
2) the multiplicity of an eigenvalue η of L_S is
$\#\{k \in \mathbb{Z} : \eta_k(S) = \eta\}$
3) for $S=0$ we have $\eta_1(0) = \eta_2(0) = \dots = \eta_{2n}(0) = 0$.

Given this convention for our ordering, we define:

Def: Let $x(t)$ be a periodic orbit with associated path of symmetric matrices $S(t)$) and operator L_S having spectrum $\eta_k(S)$. Then, the Conley-Zehnder index of x is:

$$\mu_{CZ}(x) := n - \max\{k \in \mathbb{Z} : \eta_k(S) < 0\}.$$

Examples:

1) consider a non-degenerate critical pt. x_0 that is a Minimum of $H_0 : M \to \mathbb{R}$ as a 1-periodic orbit of X_{H_0}.
Then, we seek eigenvalue η of $i\dot{\xi} - \lambda\xi = \eta\xi$, some $\lambda > 0$,
i.e. of $\dot{\xi} = -i(\lambda + \eta)\xi$ $\quad (\xi(t+1) = \xi(t))$.

these eigenvalues are then $\eta = 2\pi k - \lambda$

Which we can order as:

$$\ldots \eta_0 = 2\pi - \lambda,\ \eta_1 = -\lambda, \ldots, \eta_{2n} = -\lambda,\ \Big|\ \eta_{2n+1} = 2\pi - \lambda, \ldots$$

so that, for $0 < \lambda < 2\pi$, we have:

$$\mu_{CZ}(x_0) = -n \qquad (\|H_0\|_{C^2} << 1)$$

(which agrees with our previous claim $\mu_{CZ}(x_0) + n \overset{*}{=} \mu_{morse}(x_0)$).

2) Likewise, for a non-degenerate <u>maximum</u> of $H_0 : M \to \mathbb{R}$,

we have the ordering:

$$\ldots \eta_0 = -2\pi + \lambda \ \Big|\ < \eta_1 = \lambda = \ldots = \eta_{2n} = \lambda < \eta_{2n+1} = \lambda + 2\pi \leq \ldots$$

for some $\lambda > 0$. When $0 < \lambda < 2\pi$, we have then

$\mu_{CZ}(x_0) = n$, (which agrees with $\mu_{CZ}(x_0) + n \overset{*}{=} \mu_{Morse}(x_0)$ $\|H_0\|_{C^2} << 1$).

3) For an exercise, consider a saddle point, say in dimension $2 = 2n$,

with $H_0 = \lambda xy + \ldots$; $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

verify that the eigenvalues of $L_S \xi = \eta \xi$ have the ordering:

$$\ldots \eta_0 = -\sqrt{\lambda^2 + 4\pi^2},\ \eta_1 = -|\lambda| \ \Big|\ \eta_2 = |\lambda|,\ \eta_3 = \sqrt{\lambda^2 + 4\pi^2}, \ldots$$

So that $\mu_{CZ}(x_0) = 0$.