Online Appendix
Elections with Opinion Polls: Information Acquisition and Aggregation

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B Lemma C

In this proof, we write $x = 1/\lambda$. Define a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\psi(t) \equiv t - x\phi(t).$$

By definition, t_λ is the unique solution to equation $\psi(t) = 0$.

First, we consider the case of $x \in (0, 3)$. Note that $\frac{x}{\sqrt{2\pi}} \geq t_\lambda$ since ψ is strictly increasing and $\psi(\frac{x}{\sqrt{2\pi}}) > 0$. To show that $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > \Phi(t_\lambda)$, it suffices to show that $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > \Phi(\frac{x}{\sqrt{2\pi}})$. Define a function $l : (0, 3] \to \mathbb{R}$ as

$$l(x) = \begin{cases} \frac{x}{2\pi} + \frac{1}{2} & \text{if } x \in (0, 2) \\ \frac{x}{2\pi} - \frac{2}{2\pi} + \Phi(\frac{2}{2\pi}) & \text{if } x \in [2, 3]. \end{cases}$$

This function l is plotted in Figure 1. It suffices to show that

$$\frac{\mathsf{e}^x}{1 + \mathsf{e}^x} > l(x) > \Phi\left(\frac{x}{\sqrt{2\pi}}\right).$$

We prove these inequalities. To show that $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > l(x)$, it suffices to show that $g(x) \equiv \frac{\mathsf{e}^x}{1+\mathsf{e}^x} - l(x) > 0$. Since g is strictly convex and since $g(0) = 0$ and $g(2-), g(2), g(3) > 0$, it follows that $g(x) > 0$ for any $x \in (0, 3]$. To show that $l(x) > \Phi(\frac{x}{\sqrt{2\pi}})$, it suffices to show that $h(x) \equiv l(x) - \Phi(\frac{x}{\sqrt{2\pi}})$. Note that $h'(x) = \frac{1}{2\pi} - \frac{1}{\sqrt{2\pi}}\phi(\frac{x}{\sqrt{2\pi}}) > 0$ for any $x \neq 2$, where $\phi(\frac{x}{\sqrt{2\pi}}) < \phi(0) = \frac{1}{\sqrt{2\pi}}$. Since $h(0) = h(2) = 0$ by construction, $h(x) > 0$ for any $x \in (0, 3)$. Hence, $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > \Phi(t_\lambda)$ for any $x \in (0, 3)$.

Second, we consider the case of $x \in [3, \infty)$. Note that $\sqrt{2 \ln x} \geq t_\lambda$ since ψ is strictly increasing and $\psi(\sqrt{2 \ln x}) > 0$. To show that $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > \Phi(t_\lambda)$, it suffices to show that $\frac{\mathsf{e}^x}{1+\mathsf{e}^x} > \Phi(\sqrt{2 \ln x})$. Let $\Psi(t) \equiv 1 - \phi(t)(t^{-1} - t^{-3})$, and then we have $\Psi(t) > \Phi(t)$ for any $t > 0$.[1] This is because

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[1]Note the following asymptotic expansion: $\Phi(t) = 1 - \phi(t)\sum_{n=0}^{\infty}(-1)^n(2n-1)!!t^{-2n-1}$, where $(2n-1)!!$ is the double factorial of $2n - 1$. We obtain Ψ by truncating the expansion at $n = 1$.

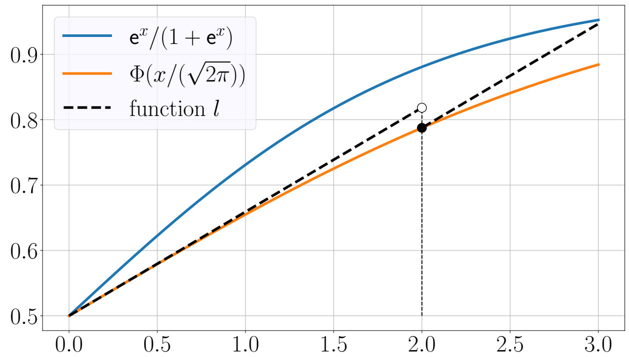

Figure 1: the function $l(x)$

$\frac{d}{dt}(\Psi(t) - \Phi(t)) = -3\phi(t)t^{-4} < 0$ and $\lim_{t\to\infty} \Psi(t) - \Phi(t) = 0$. To show that $\frac{e^x}{1+e^x} > \Phi(t_\lambda)$, it suffices to prove that $\frac{e^x}{1+e^x} > \Psi(\sqrt{2\ln x})$. By algebra, it is equivalent to

$$\frac{(2\ln x - 1)^2}{(2\ln x)^3} > \frac{2\pi x^2}{(1+e^x)^2}.$$

Since

$$\frac{(2\ln x - 1)^2}{(2\ln x)^3} = \frac{(1 - 1/(2\ln x))^2}{2\ln x} \geq \frac{(1 - 1/(2\ln 3))^2}{2\ln x},$$

it suffices to show that

$$\frac{(1 - 1/(2\ln 3))^2}{2\ln x} > \frac{2\pi x^2}{(1+e^x)^2}.$$

By Taylor expansion, $e^x \geq e^3(1 + (x-3) + (x-3)^2/2)$. Since $e^3 > 20$, it suffices to show that

$$\frac{(1 - 1/(2\ln 3))^2}{2\ln x} > \frac{2\pi x^2}{(1 + 20(1 + (x-3) + (x-3)^2/2))^2}.$$

By rearranging the terms, this inequality is equivalent to

$$(1 - 1/(2\ln 3))^2(20x - 39 + 10(x-3)^2)^2 > 4\pi x^2 \ln x$$

For $x \geq 3$, we have $\ln \frac{x}{3} \leq \frac{x}{3} - 1$, which implies that

$$\ln x \leq \frac{x}{3} + (\ln 3 - 1) < \frac{x}{3} + \frac{1}{10}.$$

Since $(1 - \frac{1}{2\ln 3})^2 > \frac{29}{100}$ and $\pi < \frac{22}{7}$, it suffices to show that

$$\frac{29}{100}\Big(20x - 39 + 10(x-3)^2\Big)^2 > \frac{88}{7}x^2\left(\frac{x}{3} + \frac{1}{10}\right),$$

which is simplified to

$$29x^4 - \frac{4960}{21}x^3 + \frac{26549}{35}x^2 - \frac{5916}{5}x + \frac{75429}{100} > 0.$$

To show this inequality, let $\mathtt{p}(x)$ denote the polynomial on the LHS. Since $\mathtt{p}$ is strictly convex and $\mathtt{p}'(2) = -\frac{1936}{35}$ and $\mathtt{p}'(3) = \frac{4302}{35}$, it is strictly increasing on $(3, \infty)$. Then, $\mathtt{p}(x) > 0$ for any $x \geq 3$ since $\mathtt{p}(3) = \frac{2403}{700} > 0$. Therefore, $\frac{\mathrm{e}^x}{1+\mathrm{e}^x} > \Phi(t_\lambda)$ for any $x \in [3, \infty)$.

C Theorems 1′ and 2′

We extend all results of elections with opinion polls (Lemmas 1 to 3 and Theorems 1 and 2) to the elections with supermajority rule or unanimity rule. The proofs are mostly the same as the original proofs for the elections with simple majority rule, but we need to modify some details.

Preliminaries To simplify the presentation, we denote by $\tilde{u}$ the payoff function under supermajority rule or unanimity rule. Formally, we define each voter's payoff function $\tilde{u} : [0,1] \times \Theta \to \{0,1\}$ by, given a threshold $\alpha \in (\frac{1}{2}, 1]$,

$$\tilde{u}(\bar{a}_N, \theta) = \begin{cases} 1 & \text{if } \mathbb{1}\{\bar{a}_N \geq \alpha\} = \theta \\ 0 & \text{if } \mathbb{1}\{\bar{a}_N \geq \alpha\} \neq \theta, \end{cases}$$

where alternative 1 is chosen if and only if the vote share $\bar{a}_N$ is at least the threshold α; namely, the chosen alternative is $\mathbb{1}\{\bar{a}_N \geq \alpha\}$.

Next, we define the integer $\tilde{n}$ such that alternative 1 is chosen if and only if it receives at least $\tilde{n}+1$ votes. That is, we have $\tilde{n} = k$ for the unique integer k such that $\frac{k}{N} < \alpha \leq \frac{k+1}{N}$.[2] In particular, we have $\tilde{n} = N - 1$ under the unanimity rule.

C.1 Theorem 1′

Lemma 1′. *In any election $\tilde{\mathcal{P}}_N$, every symmetric equilibrium $\tilde{P}^*_N$ has some $\tilde{p}^*_N \in [0,1]$ such that for each θ and each $k = 0, 1, \ldots, N$, the equilibrium vote share $\bar{a}_N$ satisfies*

$$\Pr\left(\bar{a}_N = \frac{k}{N} \mid \theta\right) = \frac{1}{\tilde{Z}_N(\tilde{p}^*_N, \theta)} \binom{N}{k} \exp\left(\frac{\tilde{u}(\frac{k}{N}, \theta)}{\lambda}\right) (\tilde{p}^*_N)^k (1 - \tilde{p}^*_N)^{N-k}, \tag{1}$$

where $\tilde{Z}_N : [0,1] \times \Theta \to \mathbb{R}$ is the function defined by

$$\tilde{Z}_N(p, \theta) = \sum_{k=0}^{N} \binom{N}{k} \exp\left(\frac{\tilde{u}(\frac{k}{N}, \theta)}{\lambda}\right) p^k (1-p)^{N-k}, \tag{2}$$

and $\tilde{p}^*_N \in [0,1]$ is the unconditional probability of each voter voting for alternative 1.

[2] In the baseline election under the simple majority rule with $N = 2n + 1$ voters, we have $\tilde{n} = n$.

One of the following holds:

1. *$\tilde{P}_N^*$ is an uninformative equilibrium if and only if $\tilde{p}_N^* \in \{0,1\}$.*
2. *$\tilde{P}_N^*$ is an informative equilibrium if and only if $\tilde{p}_N^* \in (0,1)$ is a solution to equation*

$$\frac{\tilde{Z}_N(p,1)}{\tilde{Z}_N(p,0)} = \frac{\mu(1)}{\mu(0)}. \tag{3}$$

Proof. The original proof goes through up to deriving the first-order condition (9) if each u is replaced by $\tilde{u}$.[3] Here is the modified first-order condition:

$$\sum_\theta \mu(\theta) \cdot \frac{\sum_{k=0}^{2n} \binom{2n}{k} \left[\exp\left(\frac{\tilde{u}(\frac{k+1}{N}, \theta)}{\lambda} \right) - \exp\left(\frac{\tilde{u}(\frac{k}{N}, \theta)}{\lambda} \right) \right] (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{2n-k}}{\sum_{k=0}^{N} \binom{N}{k} \exp\left(\frac{\tilde{u}(\frac{k}{N}, \theta)}{\lambda} \right) (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{N-k}} = 0.$$

The denominator is $\tilde{Z}_N(\tilde{p}_N^*, \theta)$ by the definition of (2). In the numerator, if $k \neq \tilde{n}$ then the square bracket is zero, while if $k = \tilde{n}$ then the square bracket equals $\mathsf{e}^{1/\lambda} - 1$ when $\theta = 1$ and $1 - \mathsf{e}^{1/\lambda}$ when $\theta = 0$. Substituting them into (9), we have

$$\frac{\mu(1)}{\tilde{Z}_N(\tilde{p}_N^*, 1)} \binom{2n}{\tilde{n}} (\tilde{p}_N^*)^{\tilde{n}} (1-\tilde{p}_N^*)^{2n-\tilde{n}} (\mathsf{e}^{1/\lambda} - 1) + \frac{\mu(0)}{\tilde{Z}_N(p_N^*, 0)} \binom{2n}{\tilde{n}} (\tilde{p}_N^*)^{\tilde{n}} (1-\tilde{p}_N^*)^{2n-\tilde{n}} (1 - \mathsf{e}^{1/\lambda}) = 0,$$

which is equivalent to (3). For $\tilde{p}_N^* \in (0,1)$, $(\tilde{p}_N^*, \ldots, \tilde{p}_N^*)$ is a Nash equilibrium if and only if $\tilde{p}_N^*$ is a solution to (3). ■

Lemma 2′. *In any election $\tilde{\mathcal{P}}_N$, Lemma 2 holds as is.*

Proof. The original proof goes through with a few modifications. We define the function $\tilde{W}_N : [0,1] \times \Theta \to \mathbb{R}$ by

$$\begin{aligned} \tilde{W}_N(p,1) &= \sum_{k=\tilde{n}+1}^{N} \binom{N}{k} p^k (1-p)^{N-k}, \\ \tilde{W}_N(p,0) &= \sum_{k=0}^{\tilde{n}} \binom{N}{k} p^k (1-p)^{N-k}. \end{aligned} \tag{4}$$

This function plays the same role as the function W plays in the original proof. Note that $\tilde{W}_N(p,1) + \tilde{W}_N(p,0) = 1$ and that $\tilde{W}_N(p,1)$ is strictly increasing in p and $\tilde{W}_N(p,0)$ is strictly decreasing in p.

Then, it holds that

$$\begin{aligned} \tilde{Z}_N(p,1) &= \tilde{W}_N(p,0) + \mathsf{e}^{1/\lambda} \tilde{W}_N(p,1), \\ \tilde{Z}_N(p,0) &= \mathsf{e}^{1/\lambda} \tilde{W}_N(p,0) + \tilde{W}_N(p,1). \end{aligned} \tag{5}$$

[3] Lemma A remains true if each u is replaced by $\tilde{u}$.

Note that $\frac{\tilde{Z}_N(p,1)}{\tilde{Z}_N(p,0)}$ is continuous and strictly increasing in p, because $\tilde{Z}_N(p,1)$ is strictly increasing in p and $\tilde{Z}_N(p,0)$ strictly decreasing in p.

The remaining argument is the same as Step 2 in the original proof with the modification that the functions W_N and Z_N are replaced by $\tilde{W}_N$ and $\tilde{Z}_N$, respectively. ■

Proof of Theorem 1′. This proof is analogous to the original proof. We only need to repeat the same argument by setting the winning threshold α and replacing the functions u and W with $\tilde{u}$ and $\tilde{W}$ respectively and the integer n with $\tilde{n}$. Then, we have

$$\Pr(\bar{a}_N \geq \alpha \mid \theta = 1) = \frac{\mathrm{e}^{1/\lambda} - \frac{\mu(0)}{\mu(1)}}{\mathrm{e}^{1/\lambda} - \mathrm{e}^{-1/\lambda}},$$
$$\Pr(\bar{a}_N < \alpha \mid \theta = 0) = \frac{\mathrm{e}^{1/\lambda} - \frac{\mu(1)}{\mu(0)}}{\mathrm{e}^{1/\lambda} - \mathrm{e}^{-1/\lambda}}.$$

Hence,

$$\Pr(\tilde{u}(\bar{a}_N, \theta) = 1) = \mu(1)\Pr(\bar{a}_N \geq \alpha \mid \theta = 1) + \mu(0)\Pr(\bar{a}_N < \alpha \mid \theta = 1) = \frac{\mathrm{e}^{1/\lambda}}{1 + \mathrm{e}^{1/\lambda}},$$

as desired. The proof that $\frac{\mathrm{e}^{1/\lambda}}{1+\mathrm{e}^{1/\lambda}} > \max\{\mu(1), \mu(0)\}$ when the informative equilibrium exists is exactly the same as in the original proof. ■

C.2 Proof of Theorem 2′

Lemma 3′. *For each N, let P_N^* be the informative equilibrium of any election $\tilde{\mathcal{P}}_N$ that satisfies Condition 1. Then,*

$$\lim_{N\to\infty} p_N^* = \alpha.$$

Proof. First, we consider the case of the supermajority rule, with a winning threshold $\alpha \in (\frac{1}{2}, 1)$. It suffices to show that for any small $\epsilon > 0$ such that $0 < \alpha - \epsilon < \alpha + \epsilon < 1$, if we have a sufficiently large N then

$$\frac{\tilde{Z}_N(\alpha-\epsilon, 1)}{\tilde{Z}_N(\alpha-\epsilon, 0)} < \frac{\mu(1)}{\mu(0)} < \frac{\tilde{Z}_N(\alpha+\epsilon, 1)}{\tilde{Z}_N(\alpha+\epsilon, 0)}. \tag{6}$$

To see that (6) is sufficient, we note that $\frac{\tilde{Z}_N(p,1)}{\tilde{Z}_N(p,0)}$ is continuous and strictly increasing in p. If (6) is true then $\tilde{p}_N^* \in (\alpha - \epsilon, \alpha + \epsilon)$, where $\tilde{p}_N^*$ is a solution to (3).

We show auxiliary inequalities. For any $\delta > 0$, there is an N_δ such that for any $N > N_\delta$,

$$\begin{aligned} &\tilde{W}_N(\alpha+\epsilon, 1) > 1 - \delta, \quad \tilde{W}_N(\alpha+\epsilon, 0) < \delta, \\ &\tilde{W}_N(\alpha-\epsilon, 0) > 1 - \delta, \quad \tilde{W}_N(\alpha-\epsilon, 1) < \delta, \end{aligned} \tag{7}$$

where $\tilde{W}_N$ is defined in (4). To see these inequalities, let $\tilde{w}_1, \ldots, \tilde{w}_N$ be i.i.d. Bernoulli random variables that take values 1 and 0 with probabilities $\alpha+\epsilon$ and $1-\alpha-\epsilon$ respectively. Then, $\tilde{W}_N(\alpha+\epsilon, 1)$ and $\tilde{W}_N(\alpha+\epsilon, 0)$ are the probabilities that the sample average $\frac{1}{N}\sum_{i=1}^N \tilde{w}_i$ is, respectively, strictly greater than α and strictly less than α. By the law of large numbers, there is an N'_δ such that for any $N > N'_\delta$, we have $\tilde{W}_N(\alpha+\epsilon, 1) > 1-\delta$ and $\tilde{W}_N(\alpha+\epsilon, 0) < \delta$. To see the other two inequalities, let $\tilde{w}'_1, \ldots, \tilde{w}'_N$ be i.i.d. Bernoulli random variables that take values 1 and 0 with probabilities $\alpha-\epsilon$ and $1-\alpha+\epsilon$ respectively. By the same argument, there is an N''_δ such that for any $N > N''_\delta$, we have $W_N(\alpha-\epsilon, 0) > 1-\delta$ and $W_N(\alpha-\epsilon, 1) < \delta$. Lastly, let $N_\delta = \max\{N'_\delta, N''_\delta\}$.

Now we prove (6). This step is the same as that in the original proof with the modification that the functions W_N and Z_N are replaced by $\tilde{W}_N$ and $\tilde{Z}_N$ respectively and that the winning threshold $\frac{1}{2}$ is replaced by α.

Second, we consider the case of the unanimity rule, with a winning threshold $\alpha = 1$. It suffices to show that for any small $\epsilon > 0$, if N is sufficiently large,

$$\frac{\tilde{Z}_N(1-\epsilon, 1)}{\tilde{Z}_N(1-\epsilon, 0)} < \frac{\mu(1)}{\mu(0)}. \tag{8}$$

To see that (8) is sufficient, we note that $\frac{\tilde{Z}_N(p,1)}{\tilde{Z}_N(p,0)}$ is continuous and strictly increasing in p. If (8) is true then $\tilde{p}^*_N > 1-\epsilon$, where $\tilde{p}^*_N$ is a solution to (3).

We show auxiliary inequalities. For any $\delta > 0$, there is an N_δ such that for any $N > N_\delta$,

$$\tilde{W}_N(1-\epsilon, 1) < \delta, \quad \tilde{W}_N(1-\epsilon, 0) > 1-\delta, \tag{9}$$

where $\tilde{W}_N$ is defined in (4). To see these inequalities, let $\tilde{w}_1, \ldots, \tilde{w}_N$ be i.i.d. Bernoulli random variables that take values 1 and 0 with probabilities $1-\epsilon$ and ϵ respectively. Then, $\tilde{W}_N(1-\epsilon, 1)$ and $\tilde{W}_N(1-\epsilon, 0)$ are the probabilities that the sample average $\frac{1}{N}\sum_{i=1}^N \tilde{w}_i$ is, respectively, equal to 1 and strictly less than 1. By the law of large numbers, there is an N_δ such that for any $N > N_\delta$, we have $\tilde{W}_N(1-\epsilon, 1) < \delta$ and $\tilde{W}_N(1-\epsilon, 0) > 1-\delta$.

We show another inequality. Under Condition 1, there exists a small $\delta > 0$ such that

$$\frac{1+\mathsf{e}^{1/\lambda}\delta}{\mathsf{e}^{1/\lambda}(1-\delta)} < \frac{\mu(1)}{\mu(0)}. \tag{10}$$

To prove this inequality, note that for a small enough δ, we have the LHS arbitrarily close to $\mathsf{e}^{-1/\lambda}$. Since $\mathsf{e}^{-1/\lambda} < \frac{\mu(1)}{\mu(0)}$ (Condition 1), we obtain (10).

Now we prove (8). For any $N > N_\delta$,

$$\frac{\tilde{Z}_N(1-\epsilon, 1)}{\tilde{Z}_N(1-\epsilon, 0)} = \frac{\tilde{W}_N(1-\epsilon, 0) + \mathsf{e}^{1/\lambda}\tilde{W}_N(1-\epsilon, 1)}{\mathsf{e}^{1/\lambda}\tilde{W}_N(1-\epsilon, 0) + \tilde{W}_N(1-\epsilon, 1)} < \frac{1+\mathsf{e}^{1/\lambda}\delta}{\mathsf{e}^{1/\lambda}(1-\delta)} < \frac{\mu(1)}{\mu(0)},$$

where the equality is by (5), the first inequality by (9), and the second inequality by (10). Thus, we have (8), which completes the proof. ∎

Proof of Theorem 2′. Fix any θ and any l, h such that $0 \le l < h \le 1$. As in the original proof,

$$\frac{1}{N} \ln \Pr(\bar{a}_N \in [l,h] \mid \theta) = \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([l,h],\theta) - \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1],\theta), \tag{11}$$

where for any interval $T \subset [0,1]$,

$$\tilde{\mathcal{Z}}_N(T,\theta) \equiv \sum_{k:\frac{k}{N}\in T} \binom{N}{k} \exp\left(\frac{\tilde{u}(\frac{k}{N},\theta)}{\lambda}\right) (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{N-k}.$$

First, we consider the case of the supermajority rule, with a winning threshold $\alpha \in (\frac{1}{2}, 1)$. Fix any $\delta > 0$. As in the original proof, there exists an N_1 such that for any $N \ge N_1$ and any k,

$$-N\delta < \frac{\tilde{u}(\frac{k}{N},\theta)}{\lambda} < N\delta,$$

and by Lemma 3′, there exists an N_2 such that for any $N \ge N_2$ and any k,

$$\mathsf{e}^{-N\delta}\alpha^k(1-\alpha)^{N-k} < (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{N-k} < \mathsf{e}^{N\delta}\alpha^k(1-\alpha)^{N-k}.$$

By these inequalities, for any $N \ge \max\{N_1, N_2\}$,

$$\left| \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([l,h],\theta) - \frac{1}{N} \ln \sum_{k:\frac{k}{N}\in[l,h]} \binom{N}{k} \alpha^k (1-\alpha)^{N-k} \right| < 2\delta. \tag{12}$$

By Sanov's theorem, there exists an N_3 such that for any $N \ge N_3$,

$$\left| \frac{1}{N} \ln \sum_{k:\frac{k}{N}\in[l,h]} \binom{N}{k} \alpha^k (1-\alpha)^{N-k} + \min_{t\in[l,h]} \{D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(\alpha))\} \right| < \delta, \tag{13}$$

where $D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(\alpha))$ is the Kullback–Leibler divergence of $\mathsf{B}(t)$ from $\mathsf{B}(\alpha)$. Note the non-negativity property: $D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(\alpha)) \ge 0$ for all $t \in [0,1]$, with equality if and only if $t = \alpha$. The minimum exists in (13) since $D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(\alpha))$ is continuous in t and $[l,h]$ is compact. The remaining argument (corresponding to Steps 3 and 4 in the original proof) is the same as that in the original proof with the modification that the winning threshold $\frac{1}{2}$ is replaced by α.

Second, we consider the case of the unanimity rule, with a winning threshold $\alpha = 1$. Fix any $\delta > 0$. There exists an N_1 such that $\tilde{u}(\frac{k}{N},\theta)/\lambda > -N\delta$ for any $N \ge N_1$. Hence,

$$\tilde{\mathcal{Z}}_N([0,1],\theta) \ge \exp\left(\frac{\tilde{u}(1,\theta)}{\lambda}\right)(\tilde{p}_N^*)^N > \mathsf{e}^{-N\delta}(\tilde{p}_N^*)^N.$$

Since $\lim_{N\to\infty} \tilde{p}_N^* = 1$ (Lemma 3′),

$$\varliminf_{N\to\infty} \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1],\theta) \ge -\delta.$$

Since the choice of $\delta > 0$ is arbitrary,

$$\lim_{N\to\infty} \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1],\theta) \geq 0.$$

Since $\tilde{u}(\frac{k}{N},1) = 0$ and $\tilde{u}(\frac{k}{N},0) = \mathsf{e}^{1/\lambda}$ for all $k \neq N$,

$$\begin{aligned}\tilde{\mathcal{Z}}_N([0,1-\epsilon],\theta) &= \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} \exp\left(\frac{\tilde{u}(\frac{k}{N},\theta)}{\lambda}\right) (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{N-k} \\ &\leq \mathsf{e}^{1/\lambda} \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} (\tilde{p}_N^*)^k (1-\tilde{p}_N^*)^{N-k}.\end{aligned}$$

Fix any $\eta \in (0,\epsilon)$. Since $\lim_{N\to\infty} \tilde{p}_N^* = 1$ (Lemma 3′), there exists an N' such that $\tilde{p}_N^* > 1-\eta$ for any $N > N'$. Since $\tilde{p}_N^* \leq 1$ and $1-\tilde{p}_N^* < \eta$,

$$\tilde{\mathcal{Z}}_N([0,1-\epsilon],\theta) < \mathsf{e}^{1/\lambda} \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} \eta^{N-k} < \mathsf{e}^{1/\lambda}(1-\eta)^{-N} \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} (1-\eta)^k \eta^{N-k}.$$

Hence,

$$\lim_{N\to\infty} \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1-\epsilon],\theta) \leq -\ln(1-\eta) + \lim_{N\to\infty} \frac{1}{N} \ln \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} (1-\eta)^k \eta^{N-k}.$$

By Sanov's theorem,

$$\lim_{N\to\infty} \frac{1}{N} \ln \sum_{k:\frac{k}{N}\leq 1-\epsilon} \binom{N}{k} (1-\eta)^k \eta^{N-k} = -\min_{t\in[0,1-\epsilon]} \{D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(1-\eta))\}.$$

Since $\eta < \epsilon$, it follows that

$$\min_{t\in[0,1-\epsilon]} \{D_{\mathsf{KL}}(\mathsf{B}(t) \parallel \mathsf{B}(1-\eta))\} = D_{\mathsf{KL}}(\mathsf{B}(1-\epsilon) \parallel \mathsf{B}(1-\eta)).$$

Then,

$$\lim_{N\to\infty} \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1-\epsilon],\theta) \leq -\ln(1-\eta) - D_{\mathsf{KL}}(\mathsf{B}(1-\epsilon) \parallel \mathsf{B}(1-\eta)).$$

The choice of $\eta > 0$ is arbitrary. Since $\lim_{\eta\to 0} \ln(1-\eta) = 0$ and $\lim_{\eta\to 0} D_{\mathsf{KL}}(\mathsf{B}(1-\epsilon) \parallel \mathsf{B}(1-\eta)) = +\infty$, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \ln \tilde{\mathcal{Z}}_N([0,1-\epsilon],\theta) = -\infty.$$

By (11),

$$\lim_{N\to\infty} \frac{1}{N} \ln \Pr(\bar{a}_N \in [0, 1-\epsilon] \mid \theta = 1) = -\infty.$$

The remaining argument (corresponding to Step 4 in the original proof) is the same as that in the original proof with $\frac{1}{2}$ replaced by α. ■