Cálculo de Probabilidades II
Respuestas Tema 2

1. Si

$$X_i \sim Ber(p) \quad y \quad Y = \sum_{i=1}^{n} x_i$$

entonces,

$$\begin{aligned} M_Y(t) &= E(E^{ty}) \\ &= E(e^{tx_1}e^{tx_2}...e^{tx_n}) \\ &= \prod_{i=1}^{n} E(e^{tx_i}) \\ &= (e^t p + (1-p)^n) \end{aligned}$$

$$Y \sim Bin(n, p)$$

2. Sea

$$X \sim \chi_k^2$$

Y

$$Y = Z_1^2 + Z_2^2 + ... + Z_n^2$$

$$\begin{aligned} \rightarrow M_Y(t) &= E(e^{ty}) \\ &= \prod_{i=1}^{n} e^{tk(z_i)^2} \\ &= (1-2t)^{nk/2} \end{aligned}$$

$$\rightarrow Y \sim \chi^2_{\sum n_i}$$

3. Sea Y = aX + b

$$\begin{aligned} \rightarrow M_Y t &= E(e^{ty}) \\ &= E(e^{t(aX+b)}) \\ &= e^{tb} M_X(at) \end{aligned}$$

4. Sea

$$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\ln x - \mu)^2/2\sigma^2}$$

$$E(X) = e^{\mu + \sigma^2/2}$$

$$Var(X) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

5. Sea X $\sim$ N(0, 1), p.d.

$$Z = X^2 \sim \chi^2_{(1)}$$

Dem.

$$\begin{aligned} M_Z(t) &= \int_0^\infty e^{-x^2(t-\frac{1}{2})} \frac{1}{\sqrt{2\pi}} \mathtt{d}x \\ &= \frac{1}{t-\frac{1}{2}} \\ &\to Z = X^2 \sim \chi^2_{(1)} \end{aligned}$$

6. Sean

$$X_i \sim Ga(\alpha, 1)$$
$$Y = \sum_{i=1}^{n} X_i$$

(a) Con n = 2

$$\begin{aligned} f_{X_1,X_2}(y) &= \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y-x_1) \mathtt{d}x_1 \\ &= \int_0^y f_{X_1}(x_1) f_{X_2}(y-x_1) \mathtt{d}x_1 \\ &= \int_0^y \frac{x_1^{\alpha-1} e^{-x_1}}{\Gamma(\alpha)} \frac{(y-x_1)^{\alpha-1} e^{-(y-x_1)}}{\Gamma(\alpha)} \mathtt{d}x_1 \end{aligned}$$

Sea

$$x_1 = yt \quad \mathtt{d}x_1 = y\mathtt{d}t$$

y sustituyendo en la integral anterior,

$$\begin{aligned} f_{X_1,X_2}(y) &= e^{-y} y^{\alpha+\alpha-1} \int_0^1 y t^{\alpha-1} (1-t)^{\alpha-1} \frac{1}{\Gamma(\alpha)\Gamma(\alpha)} \mathtt{d}t \\ &= \frac{e^{-y} y^{2\alpha-1}}{\Gamma(2\alpha)} \sim Ga(2\alpha, 1) \end{aligned}$$

(b) Sea

$$Y \sim Ga(\alpha, n) \quad y \quad W = 2nY.$$

$$\begin{aligned} P(W \leq w) &= P(2nY \leq y) \\ &= P(Y \leq \frac{w}{2n}) \\ &= F_Y(\frac{w}{2n}) \end{aligned}$$

$$\rightarrow f_W(w) = \frac{1}{2n} f_Y(\frac{w}{2n})$$
$$= \quad \frac{w^{\alpha-1}}{\Gamma(\alpha)} e^{-w/2} \frac{1}{2^{\alpha}} \sim \chi^2_{2\alpha}$$

(c) Sea

$$X \sim Ga(\alpha, 1) \quad y \quad Y = \frac{X}{n}$$

$$P(Y \leq y) = P(\frac{X}{n} \leq y)$$
$$= P(X \leq ny)$$
$$= F_X(ny)$$

$$\rightarrow f_Y(y) = n * f_X(ny)$$
$$= \frac{1}{\Gamma(\alpha)} e^{ny} y^{\alpha-1} n^{\alpha} \sim Ga(\alpha, n)$$

7. Sea

$$Y = \sum_{i=1}^{n} a_i X_i$$

P.D.

$$Y \sim N(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} (a_i \sigma_i)^2)$$

Sea

$$u_i = a_i x_i$$

$$M_U(t) = E(e^{t a_i x_i})$$
$$= exp(t a_i \mu_i + \frac{(t a_i x_i)^2}{2})$$

Sea

$$Y = \sum_{i=1}^{n} u_i$$

$$M_Y(t) = E(e^{\sum_{i=1}^{n} t u_i})$$
$$= \prod_{i=1}^{n} u_i E(e^{t u_i})$$
$$= exp(\sum_{i=1}^{n} t a_i \mu_i + \sum_{i=1}^{n} \frac{(t a_i \sigma_i)^2}{2})$$

$$\rightarrow Y \sim N(\sum_{i=1}^{n} a_i\mu_i, \sqrt{\sum_{i=1}^{n}(a_i\sigma_i)^2})$$

8. Sea $f_X(x)$ y Y = aX + b

$$\begin{aligned} F_Y &= P(Y \leq y) \\ &= P(aX + b \leq y) \\ &= P(X \leq \frac{y-b}{a}) \end{aligned}$$

$$\rightarrow f_Y = \frac{d}{dy} F_X(\frac{y-b}{a})$$

9. Sea

$$Y = e^x \qquad X \sim N(\mu, \sigma^2)$$

$$\begin{aligned} P(Y \leq y) &= P(e^x \leq y) \\ &= P(X \leq ln(y)) \\ &= F_X(ln(y)) \\ &= P(Z \leq \frac{ln(y)-\mu}{\sigma}) \\ &= \Phi(\frac{ln(y)-\mu}{\sigma}) \end{aligned}$$

Derivando obtenemos

$$\varphi_Y(y)$$

10. Sean

(a)

$$Y = (\frac{x-b}{a})^{\beta} \qquad y \quad X \sim Weibull(b, a, \beta)$$

$$\begin{aligned} P(Y \leq y) &= P((\frac{x-b}{a})^{\beta} \leq y) \\ &= P(X \leq ay^{\frac{1}{\beta}} + b) \\ &= \int_0^{ay^{\frac{1}{\beta}}+b} \frac{\beta}{a}(\frac{x}{a})^{\beta-1} e^{-(x/a)^{\beta}} \mathtt{d}x, \quad Sea\beta = 1 \quad yb = 0 \\ &= 1 - e^{-y} \end{aligned}$$

$$\rightarrow Y \sim Exp(1)$$

(b) Sean

$$Y = (\frac{x-b}{a})^{\beta} \qquad yX = ay^{\frac{1}{\beta}} + b$$

$$\begin{aligned} P(X \leq x) &= P(ay^{1/\beta} + b \leq x) \\ &= P(Y \leq (\frac{x-b}{a})^{\beta}) \\ &= 1 - exp((\frac{x-b}{a})^{\beta}) \end{aligned}$$

$$\rightarrow f_X = \frac{\beta}{a}(\frac{x-b}{a})^{\beta-1} exp((\frac{x-b}{a})^{\beta})$$
$$X \sim Weibull(a, b, \beta)$$

11. Sean

$$X \sim CauchyEstandar \quad W = \frac{1}{X}$$

$$\begin{aligned} P(W \leq w) &= P(\frac{1}{x} \leq w) \\ &= P(X \geq \frac{1}{w}) \\ &= 1 - (\frac{1}{2} + \frac{arctan\frac{1}{w}}{\pi}) \end{aligned}$$

$$f_W = \frac{1}{\pi(w^2+1)}$$
$$W \sim CauchyEstandar$$

12. Sean X ∼ U(-1, 1) W = |X|

$$P(W \leq w) = P(-w \leq X \leq w) = w$$

$$\rightarrow f_W = 1, \quad 0 \leq w \leq 1$$
$$SeaY = X^2 + 1$$
$$P(Y \leq y) = P(-\sqrt{y-1} \leq X \leq \sqrt{y-1}) = \sqrt{y-1}$$
$$f_Y = \frac{1}{2}(\frac{1}{\sqrt{y-1}})$$
$$SeaZ = \frac{1}{X+1}$$
$$P(Z \leq z) = P(X \geq \frac{1}{z} - 1) = \frac{z-1}{2z}$$

13. Sean

$$X \sim exp(1) \qquad Y = ln(X)$$

$$P(Y \leq y) = P(X \leq e^y) = 1 - e^{-e^y}$$
$$f_Y = exp(-e^y + y)$$

14. Sean

$$X \sim U(0,1) \qquad Y = e^X$$

$$P(Y \leq y) = P(X \leq ln(y)) = ln(y)$$
$$\to f_Y = \frac{1}{y} \quad I_{[1,e]}(y)$$

15. Sea

$$\theta \sim U(\frac{-\pi}{2}, \frac{\pi}{2}) \qquad R = Asen(\theta)$$

$$P(R \leq r) = P(\theta \leq arcsen(\frac{r}{a})) = \frac{arcsen(\frac{r}{a})}{\pi}$$
$$\to f_R(r) = \frac{1}{\pi A}(\frac{1}{\sqrt{1-(\frac{r}{A})^2}})$$

16. Sean

$$X \sim U(0,1) \qquad Y = -ln(X)$$
$$P(Y \leq y) = P(X \geq \frac{1}{e^y}) = 1 - \frac{1}{e^y}$$

$$\to f_Y(y) = e^{-y}$$
$$\to Y \sim Exp(1)$$

17. Sea

$$Long \sim N(3.25, 0.05^2) \quad L_1 + L_2 \sim N(6.50, .005)$$

$$\to P(L_1 + L_2 \leq 6.60) = P(Z \leq \frac{6.60 - 6.50}{\sqrt{.005}})$$
$$= \phi(1.4142)$$

18. Si X_1, X_2, X_3, ..., X_n son v.a.i.i.d.

$$Y_n = max(X_1, X_2, ..., X_n)$$
$$p.d.f_{Y_n}(y) = n(F_X(y))^{n-1} f_X(y)$$

Dem.

$$\begin{aligned} F_Y &= P(Y_n \leq y) \\ &= P(max(X_1, X_2, ..., X_n) \leq y) \\ &= \prod_{i=1}^{n} P(X_i \leq y) \\ &= (F_X(y))^n \end{aligned}$$

$$\rightarrow f_Y(y) = n(F_X(y))^{n-1} f_X(y)$$

19. Si

$$X \sim U(0,1) \quad Y_1 = minX_i$$

$$\begin{aligned} F_Y(y) &= P(Y_1 \leq y) \\ &= P(minX_i \leq y) \\ &= 1 - (1 - F_X(y))^n \end{aligned}$$

$$\rightarrow P(Y_1 \leq \frac{1}{4}) = 1 - (\frac{3}{4})^n$$

20. Si X_1 y X_2 son v.a.i.i.d. N(0,1) y sea

$$Y = \frac{(X_2 - X_1)^2}{2}$$

p.d.

$$Y \sim Ga(\frac{1}{2}, \frac{1}{2})$$

$$\begin{aligned} M_Y(t) &= E(e^{yt}) \\ &= \int_0^\infty \int_0^\infty exp(\frac{(x_2 - x_1)^2 t}{2}) exp(-\frac{1}{2}(x_1^2 x_2^2)) \mathtt{d}x_1 \mathtt{d}x_2 \\ &= \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2} \quad para \quad t < \frac{1}{2} \end{aligned}$$

21. Si

$$X_1, X_2 \quad v.a.i.i.d. \quad Normal(0,1) \quad Y_1 = X_1^2 + X_2^2 \quad Y_2 = X_2$$

$$\begin{aligned} &f(y_1, y_2) = \frac{1}{2\sqrt{y_1 - y_2^2}} \frac{1}{2\pi} e^{-y_1/2} I_{y_2^2, \infty}(y_1) I_{-\infty, \infty}(y_2) \\ &Y_1 \sim \chi_2^2 \end{aligned}$$

22. (a) Sea X_1, X_2 v.a.i.i.d. Normal(0,1)

$$Y_1 = (X_1^2 + X_2^2)^{\frac{1}{2}}$$
$$Y_2 = arctan(\frac{X_2}{X_1})$$
$$|\mathbf{J}| = \frac{y_1 sec^2(y_2)}{1 + tan^2(y_2)}$$

$$\rightarrow f(y_1, y_2) = \frac{y_1}{2\pi} e^{-y_1^2/2} I_{\Re}(y_1) I_{(0,\pi/2)}(y_2)$$

No son independientes.

(b) Sea $X_i \sim$ U(0, 1)

$$Y_1 = X_2 + X_1$$
$$Y_2 - \frac{X_1}{X_2}$$
$$|\mathbf{J}| = \frac{y_1}{(1 - y_2)^2}$$

$$\rightarrow f(y_1, y_2) = \frac{y_1}{(1 - y_2)^2} \quad I_{(0,1)}(y_2) I_{(0,1-y_2)}(y_1)$$

No son independientes.

(c) Sea $X_i \sim$ Exp(1), i = 1, 2, 3

$$Y_1 = \frac{X_1}{X_1 + X_2}$$
$$Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$$
$$Y_3 = X_1 + X_2 + X_3$$
$$|\mathbf{J}| = y_2 y_3^2$$

$$\rightarrow f(y_1, y_2, y_3) = y_2 y_3^2 e^{-y_3} I_{(0,1)}(y_1, y_2, y_3)$$

Si son independientes.

23. Sea

$$X_i \sim N(\mu_i, \sigma_i^2)$$

$$M_W(t) = E(e^{tw})$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$
$$= prod_{i=1}^{n} exp \mu_i t + \sigma_i^2 t^2$$

$$\rightarrow Y \sim N(\sum \mu_i, \sum \sigma_i^2)$$

24. (a) Sea

$$X \sim U(0,2) \quad Y \sim U(0,1)$$

$$Z = X + Y$$
$$f_Z = \frac{1}{2}z \quad I_{(0,1)}(z) + \frac{1}{2}I_{(1,2)}(z)$$

(b)

$$X, Y \sim N(0,1)$$

$$Z = \frac{Y}{X}$$
$$W = X$$
$$\rightarrow |\mathbf{J}| = W$$

$$f_{(Z,W)} = \frac{w}{2\pi}e^{-w^2(z^2+1)} \quad I_{(-\infty,\infty)}(w,z)$$
$$f_Z(z) = \int_{-\infty}^{\infty} \frac{w}{2\pi}e^{-w^2(z^2+1)}\mathtt{d}w \quad I_{(-\infty,\infty)}(z)$$
$$= \frac{1}{\pi(1+z^2)} \quad I_{\Re}(z)$$

(c) X, Y tal que

$$f_T(t) = bt^{b-1} \quad I_{(0,1)}(t)I_{(0,\infty)}(b)$$

$$Z = YX$$
$$W = Y$$
$$\rightarrow |\mathbf{J}| = \frac{1}{W}$$

$$f_{(Z,W)} = \frac{b^2}{w}z^{b-1} \quad I_{(0,1)}(z)I_{(z,1)}(w)I_{(0,\infty)}(b)$$
$$\rightarrow f_Z = b^2z^{b-1}ln(Z) \quad I_{(0,1)}(z)I_{(0,\infty)}(b)$$

25. Sea

$$f(x_i) = 2x_i \quad I_{(0,1)}(x) \qquad Z = min(X_i)$$

$$F_Z(z) = 1 - (1 - x^2)^3$$
$$\rightarrow f_Z = 6x(1 - x^2)^2$$
$$mediana = \frac{1}{\sqrt{2}}$$
$$P(Z > \frac{1}{\sqrt{2}}) = 1 - F_z(1/\sqrt{2}) = (1 - 1/2)^3 = \frac{1}{8}$$

26. (a) Sean X_1, X_2, ..., X_n v.a.i.i.d. con $F_X(x)$ y la funcion de densidad de la j-esima estadistica de orden Y_j esta dada por:

$$f(y_j) = \frac{n!}{(j-1)!(n-j)!} f_X(y)(F_X(y))^{j-1}(1 - F_X(y))^{n-j}$$

Para variables continuas.

(b) Sea $Y_1 = \min(X_i)$

$$\begin{aligned} F_{Y_1} &= P(Y_1 \leq y_1) \\ &P(min(X_i) \leq y_1) \\ &= 1 - P(min(X_i) > y_1) \\ &= 1 - \prod_{i=1}^{n} P(X_i > y_1) \\ &= 1 - (1 - F_X(y_1))^n \end{aligned}$$

$$\rightarrow f_{Y_1}(y_1) = n(1 - F_X(y_1))^{n-1} f_X(y_1)$$

(c) Sea $Y_n = \max(X_i)$

$$\begin{aligned} F_{Y_n} &= P(Y_n \leq y_n) \\ &P(max(X_i) \leq y_n) \\ &= \prod_{i=1}^{n} P(X_i \leq y_n) \\ &= (F_X(y_n))^n \end{aligned}$$

$$\rightarrow f_{Y_n}(y_n) = n(F_X(y_1))^{n-1} f_X(y_n)$$

27. (a) Sea X_1, X_2, ..., X_n una muestra aleatoria, en donde

$$f_{X_i}(x_i) \sim U(\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma)$$

Obtener el Rango muestral:

$$\begin{aligned} &Z = Y_n - Y_1 \\ &W = Y_1 \\ &\rightarrow |\mathbf{J}| = 1 \end{aligned}$$

$$f_{(Z,W)}(z, w) = n^2 (1 - \frac{w}{2\sqrt{3}\sigma})^{n-1} \frac{1}{12\sigma} (\frac{z+w}{2\sqrt{3}\sigma})^{n-1} \quad I_{(\mu-\sqrt{3}\sigma, \mu+\sqrt{3}\sigma)}(w)(I_{(0,2\sqrt{3}\sigma)}(z)$$

$$f_Z(z) = \int_{\mu-\sqrt{3}\sigma}^{\mu-\sqrt{3}\sigma} n^2 (1 - \frac{w}{2\sqrt{3}\sigma})^{n-1} \frac{1}{12\sigma} (\frac{z+w}{2\sqrt{3}\sigma})^{n-1} \mathtt{d}w \quad I_{(0,2\sqrt{3}\sigma)}(z)$$

Obtener el rango medio muestral:

$$Z = \frac{Y_1 - Y_n}{2}$$
$$W = Y_n$$
$$\rightarrow |\mathbf{J}| = 2$$

$$f_{Z,W}(z,w) = 2n^2(1 - \frac{2z-w}{2\sqrt{3\sigma}})^{n-1}\frac{1}{12\sigma}(\frac{w}{2\sqrt{3\sigma}})^{n-1} \quad I_{(\mu-\sqrt{3\sigma},\mu)}(z)I_{(\mu-\sqrt{3\sigma},2z-\mu+\sqrt{3\sigma})}(w) + I_{(2z-\mu-\sqrt{3\sigma},\mu+\sqrt{3\sigma})}(w)I_{(\mu,\mu+\sqrt{3\sigma})}(z)$$

$$\rightarrow f_Z(z) = \int_{\mu-\sqrt{3\sigma}}^{2z-\mu+\sqrt{3\sigma}} 2n^2(1 - \frac{2z-w}{2\sqrt{3\sigma}})^{n-1}\frac{1}{12\sigma}(\frac{w}{2\sqrt{3\sigma}})^{n-1}\mathtt{d}w \quad I_{(\mu-\sqrt{3\sigma},\mu)}(z) + \int_{2z-\mu-\sqrt{3\sigma}}^{\mu+\sqrt{3\sigma}} 2n^2(1 - \frac{2z-w}{2\sqrt{3\sigma}})^{n-1}\frac{1}{12\sigma}(\frac{w}{2\sqrt{3\sigma}})^{n-1}\mathtt{d}w \quad I_{(\mu,\mu+\sqrt{3\sigma})}(z)$$

(b) Sean X_1, X_2, ..., $X_n \sim \text{Exp}(\theta)$. Obtener el rando muestral:

$$Z = Y_n - Y_1$$
$$W = Y_1$$
$$\rightarrow |\mathbf{J}| = 1$$

$$f_{Z,W}(z,w) = \theta^2 n^2(1 - e^{\theta(z+w)})^{n-1}e^{\theta w(n+1)}e^{\theta z} \quad I_{(0,\infty)}(z)I_{(z,\infty)}(w)$$
$$f_Z(z) = \int_z^\infty \theta^2 n^2(1 - e^{\theta(z+w)})^{n-1}e^{\theta w(n+1)}e^{\theta z}\mathtt{d}w \quad I_{(0,\infty)}(z)$$

Obtener el Rango medio muestral:

$$Z = \frac{Y_1 - Y_n}{2}$$
$$W = Y_n$$
$$\rightarrow |\mathbf{J}| = 2$$

$$f_{Z,W}(z,w) = 2\theta^2 n^2(1 - e^{\theta(z+w)})^{n-1}e^{\theta w(n+1)}e^{\theta z} \quad I_{(0,\infty)}(z)I_{(0,2z)}(w)$$
$$f_Z(z) = \int_0^{2z} 2\theta^2 n^2(1 - e^{\theta(z+w)})^{n-1}e^{\theta w(n+1)}e^{\theta z}\mathtt{d}w \quad I_{(0,\infty)}(z)$$

28. Sea

$$L \sim N(\mu, 1)$$

(a)

$$P(\hat{Y} > 8) = 1 - \phi(4.5)$$

(b)

$$P(6.2 \leq \hat{Y} \leq 6.8) = 2\phi(0.9) - 1$$

(c)

$$\begin{aligned} &P(\mu - 0.3 \leq \hat{Y} \leq \mu + 0.3) = 0.95 \\ &\to 2\phi(0.3\sqrt{n}) - 1 = 0.95 \\ &\to n = 42.6844 \end{aligned}$$

29. Sea N = 10, $\sigma = 1$

$$\begin{aligned} &P(a \leq S^2 \leq b) = 0.90 \\ &en \quad donde \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \sim Ga(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}) \\ &\to 0.90 = \frac{(\frac{9}{2})^{9/2}}{\Gamma(\frac{9}{2})}\int_0^b x^{\frac{7}{2}}e^{-\frac{7x}{2}}\mathtt{d}x - \frac{(\frac{9}{2})^{9/2}}{\Gamma(\frac{9}{2})}\int_0^a x^{\frac{7}{2}}e^{-\frac{7x}{2}}\mathtt{d}x \end{aligned}$$

30. Sea

$$N_1 = 6 \quad N_2 = 10$$

ambas con la misma varianza de poblacional.

$$P(\frac{S_1^2}{S_2^2} \leq b) = 0.90$$

sabemos que

$$\begin{aligned} &\frac{S_1^2}{S_2^2} \sim F(5,9) \\ &\to b = 2.611 \end{aligned}$$

31. Sea

$$P(X \leq Q_1) = 0.25, \quad X \sim N(0,1)$$

$$\begin{aligned} &Qx_1 = 1 - \phi(0.68) \\ &Qx_2 = 0 \\ &Qx_3 = \phi(0.68) \end{aligned}$$

Sea

$$Y \sim t - student(10)$$

$$\begin{aligned} &Qy_1 = 1 - F_X(.700) \\ &Qy_2 = 0.129 \\ &Qy_3 = 0.700 \end{aligned}$$

$$W \sim \chi^2(20)$$

$$Qw_1 = 15425$$
$$Qw_2 = 19337$$
$$Qw_3 = 23828$$

32. Sean

$$X \sim U(0,1) \qquad Y = ln(\frac{x}{1-x})$$

$$P(Y \leq y) = P(ln(\frac{x}{1-x} \leq y)$$
$$= P(x \leq \frac{e^y}{1+e^y})$$

$$\rightarrow f_Y(y) = \frac{e^y}{(1+e^y)^2}$$

33. Sea

$$X_i \sim \chi_2^2$$

$$W = \frac{X_1 - X_2}{2}$$
$$Z = X_1$$
$$\rightarrow |\mathbf{J}| = 2$$

$$\rightarrow f_{Z,W}(z,w) = e^{-z-w} \quad I_{(0,\infty)}(w) I_{(2w,\infty)}(z)$$
$$\rightarrow f_W(w) = e^{-3w} \quad I_{(0,\infty)}(w)$$

34. Sean

$$X \sim N(\mu, \sigma^2) \quad Y = \frac{x-\mu}{\sigma} \sim N(0,1) \quad W = Y^2$$

$$\rightarrow f_W(w) = \frac{1}{\sqrt{2\pi}} w^{\frac{1}{2}} e^{-\frac{1}{2w}} \quad I_{(0,\infty)}(w) \quad como \sqrt{\pi} = \Gamma(\frac{1}{2})$$
$$= \frac{(\frac{1}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} w^{\frac{1}{2}} e^{-\frac{1}{2w}} \quad I_{(0,\infty)}(w)$$

$$\rightarrow W \sim Ga(\frac{1}{2}, \frac{1}{2}) = \chi_1^2$$

35. Sea

$$X \sim Ga(\alpha, 1)$$
$$y_i = \frac{x_i}{Y_{k+1}}$$
$$Y_{k+1} = \sum_{i=1}^{k+1} x_i$$

sea

$$x_1 = y_1 * Y_{k+1}, x_2 = y_2 * Y_{k+1}, ..., x_k = y_k * Y_{k+1}$$

Y

$$x_{k+1} = (1 - \sum_{i=1}^{k} y_i) * y_{k+1}$$

$$\rightarrow |\mathbf{J}| = |y_{k+1}^k - 2(y_{k+1}^k * \sum y_i)|$$

$$\rightarrow f_Y(y) = (\frac{1}{\Gamma(\alpha)})^{k+1} \int_0^\infty (y_{k+1}^k - 2(y_{k+1}^k * \sum y_i)) * e^{-y_{k+1}} y_{k+1}^{2(\alpha-1)} (1 - \sum y_i)^{\alpha-1} \prod_{i=1}^{k} y_i^{\alpha-1} \mathtt{d}y_{k+1}$$

Sea

$$\gamma = (\frac{1}{\Gamma(\alpha)})^{k+1} (1 - 2\sum y_i)(1 - \sum y_i)^{\alpha-1} \prod_{i=1}^{k} y_i^{\alpha-1}$$

$$\rightarrow \gamma \int_0^\infty y_{k+1}^k e^{-y_{k+1}} \mathtt{d}y_{k+1} y_{k+1}^{2(\alpha-1)} = \gamma * \Gamma(2\alpha + k - 1)$$

$$\rightarrow f_Y(y) = (\frac{1}{\Gamma\alpha})^{k+1} (1 - 2\sum y_i)(1 - \sum y_i)^{\alpha-1} \prod_{i=1}^{k} y_i^{\alpha-1} * \Gamma(2\alpha + k - 1)$$

Con K = 1 se tiene una Beta

36. (a) Sean

$$Sea X_1, X_2, ..., X_n v.a.i. N(\mu, \sigma)$$
$$Y S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\rightarrow M = \sum (x_i - \mu)^2(t) = M \sum (x_i - \bar{x})^2(t) * M(n\bar{x} - n\mu)^2$$

Si son independientes,

$$\rightarrow M \sum (x_i - \bar{x})^2(t) = \frac{M \sum (x_i - \mu)^2(t)}{M(n\bar{x} - n\mu)^2}$$

$$\rightarrow M \sum (\frac{x_i - \bar{x}}{\sigma})^2(t) = \frac{M \sum (\frac{x_i - \mu}{\sigma})^2(t)}{M(\frac{n\bar{x} - n\mu}{\sigma})^2}$$

En donde

$$M \sum (\frac{x_i - \mu}{\sigma})^2(t) \sim \chi_n^2$$
$$M(\frac{n\bar{x} - n\mu}{\sigma})^2 \sim \chi_1^2$$

$$\rightarrow M\sum(\frac{x_i-\bar{x}}{\sigma})^2(t) = \frac{(\frac{\frac{1}{2}}{\frac{1}{2}-t})^{\frac{n}{2}}}{(\frac{\frac{1}{2}}{\frac{1}{2}-t})^{\frac{1}{2}}}$$
$$= (\frac{\frac{1}{2}}{\frac{1}{2}-t})^{\frac{n-1}{2}}$$

Por Teorema de unicidad.

$$\sum(\frac{x_i-\bar{x}}{\sigma})^2 = \frac{(n-1)S^2}{\sigma^2} \quad \sim \chi^2_{n-1}$$

(b) Sabemos que si

$$X \sim N(0,1) \quad Y \sim \chi^2_n$$

entonces

$$T = \frac{X}{\sqrt{\frac{Y}{n}}} \sim t(n)$$

$$\rightarrow \frac{\sqrt{n}(\bar{x}-\mu)}{S_X} = \frac{\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}} \sim \sqrt{\frac{\chi^2_{n-1}}{n-1}}} \sim t(n-1)$$

37. Sea n = 3

(a)

$$P(Y_1 > median) \quad median = \frac{1}{\sqrt{2}}$$

$$P(Y_1 > median) = \int_0^{\frac{1}{\sqrt{2}}} 6(1-y^2)^2 y_1 \mathtt{d}y_1$$
$$= 0.125$$

(b) Sean

$$Z_1 = \frac{Y_1}{Y_2}$$
$$Z_2 = \frac{Y_2}{Y_3}$$
$$Z_3 = Y_3$$
$$\rightarrow |J| = z_2 z_3^2$$
$$\rightarrow f(z_1,z_2,z_3) = z_2 z_3^2 f_{Y_1}(z_1 z_2 z_3) f_{Y_2}(z_2 z_3) f_{Y_3}(z_3)$$

Son independientes

38. Sea

$$f_{X,Y}(x,y) = \frac{12}{7}x(x+y)I_{(0,1)}(x)I_{(0,1)}(y)$$
$$U = min(X,Y) \quad V = min(X,Y)$$

$$\begin{aligned} F_{U,V} &= P(U \leq u, V \leq v) \\ &= P(V \leq v) - P(U > u, V \leq v) \end{aligned}$$

$$\begin{aligned} P(V \leq v) &= \int_0^v \int_0^v \frac{12}{7}x(x+y)\mathtt{d}x\mathtt{d}y \\ &= v^4 \end{aligned}$$

$$\begin{aligned} \rightarrow P(U > u, V \leq v) &= \int_u^v \int_u^v \frac{12}{7}x(x+y)\mathtt{d}y\mathtt{d}x \\ &= v^4 + u^4 - \frac{6}{7}u^2v^2 - \frac{4}{7}v^3u - \frac{4}{7}u^3v \end{aligned}$$

$$f_{U,V}(u,v) = \frac{1}{7}(4v^3u + 4u^3v - 7u^4 + 6v^2u^2) \quad I(0 < u < v < 1)$$

39. Si X_1, X_2, X_3, ..., X_n es una muestra aleatoria.

$$E(X_i) = \mu \quad Var(X_i) = \sigma^2$$
$$V(\bar{x}) = V(\sum_{i=1}^{n} \frac{1}{n}x_i) = \frac{\sigma^2}{n}$$

$$\begin{aligned} E(S^2) &= E(\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2) \\ &= \frac{1}{n-1}(\sum_{i=1}^{n} Var(x_i) - nVar(\bar{x})) \end{aligned}$$
$$\rightarrow E(S^2) = \sigma^2$$

40. Sea

$$Y = a + bx$$

$$\begin{aligned} \rho_{X,Y} &= \frac{Cov(X,Y)}{\sigma_Y \sigma_X} \\ &= \frac{bV(X)}{b\sigma_X \sigma_X} \\ &= 1 \end{aligned}$$

41. Si X_1, X_2, X_3, ..., X_n es una muestra con $V(X_i) = \sigma^2$. P.D. $Cov(x_i - \bar{x}, \bar{x}) = 0$.

$$\begin{aligned} Cov(x_i - \bar{x}, \bar{x}) &= Cov(x_i, \bar{x}) - Var(\bar{x}) \\ &= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \\ &= 0 \end{aligned}$$

42. Sea X_1, X_2, X_3, ..., X_n una sucesion de v.a.i.i.d. con media μ y varianza σ^2.

(a)

$$Nm \sim Po(\gamma)$$

$$\begin{aligned} E(S_N) &= E(E(N|S_N)) \\ &= \frac{n\mu}{\gamma} \end{aligned}$$

$$\begin{aligned} V(S_N) &= V(E(N|S_N)) + E(V(N|S_N)) \\ &= \frac{n}{\gamma^2}(n\sigma^2 + \mu) \end{aligned}$$

(b)

$$N \sim Geo(\frac{1}{\gamma}) \quad \gamma = \frac{1-p}{p}$$

$$\begin{aligned} E(S_N) &= E(E(N|S_N)) \\ &= n\mu\frac{1}{\gamma} \end{aligned}$$

$$\begin{aligned} V(S_N) &= V(E(N|S_N)) + E(V(N|S_N)) \\ &= (n\sigma\frac{1}{\gamma})^2 + \frac{1}{\gamma}(1 - \frac{1}{\gamma})n\mu \end{aligned}$$

43. Sea X una v.a. con segundo momento finito.

$$\begin{aligned} E((X-a)^2) &= E(X^2) - 2aE(X) + a^2 + 2aE(X) - a^2 \\ &= E(X^2) - E^2(X) \\ &= Var(X) \end{aligned}$$

44. Sean

$$Z \sim N(0,1) \quad Y = a + bz + cz^2$$

$$\begin{aligned} \rightarrow \rho_{Y,Z} &= \frac{Cov(Z,Y)}{\sigma_Z \sigma_Y} \\ &= \frac{b}{\sqrt{b^2 + 2c^2}} \end{aligned}$$

45.

$$\begin{aligned} V(\sum_{i=1}^{k}\alpha_i x_i) &= E[(\sum_{i=1}^{k}\alpha_i x_i)^2] - E^2[\sum_{i=1}^{k}\alpha_i x_i] \\ &= E[\sum_{i=1}^{k}\alpha_i x_i - \sum_{i=1}^{n}\alpha_i \mu_i] \\ &= E[\sum_{i=1}^{k}\alpha_i (x_i - \mu_i)^2] \\ &= E[\sum_{i=1}^{k}\alpha_i^2 Var(x_i) + 2\sum_{i<j}\alpha_i\alpha_j cov(x_i, x_j)] \end{aligned}$$

46. Sean

$$Z \sim N(0,1) \quad Y = a + bz + cz^2$$

$$\rho = \frac{Cov(Z,Y)}{\sigma_Y \sigma_Z}$$

$$\begin{aligned} Cov(Z,Y) &= E(ZY) - E(Z)E(Y) \\ &= E(ZY) \\ &= E(az + bz^2 + cz^3) \\ &= b + cE(Z^3) \\ &= b \quad por \quad independencia \end{aligned}$$

$$\begin{aligned} Var(Z) &= Var(a + bz + cz^2) \\ &= b^2 + 2c^2 \end{aligned}$$

$$\rightarrow \sigma_Y = \sqrt{b^2 + 2c^2}$$

$$\rho_{X,Y} = \frac{b}{\sqrt{b^2 + 2c^2}}$$

47. Sea X $\sim$ N(0, 1) y sea I otra v.a.i. de X tal que P(I = 1) = P(I = 0) = $\frac{1}{2}$. Y sea tambien Y = X si I = 1 y Y = -X si I = 0.

$$f_Y = \begin{cases} Y \sim N(0,1) & \text{c.p. } \frac{1}{2} \\ Y \sim N(0,1) & \text{c.p. } \frac{1}{2} \end{cases}$$

$$\rightarrow f_Y = \frac{1}{2\pi} e^{\frac{-x^2}{2}} \sim N(0,1)$$

$$Cov(X,Y) = 0$$

48. Sean

$$f(Y|X) = Bin(n,x) \quad X \sim U(0,1)$$

$$f_{X,Y}(x,y) = (\overset{n}{y})x^y(1-x)^{n-y} \quad I_{(0,1)}(x)I_{0,1,2,\ldots,n}(y)$$

$$\begin{aligned} f_{X,Y}(x,y) &= (\overset{n}{y})\int_0^1 x^y(1-x)^{n-y}\mathtt{d}x \quad I_{0,1,2,\ldots,n}(y) \\ &= (\overset{n}{y})\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \quad I_{0,1,2,\ldots,n}(y) \end{aligned}$$

* Kernel de una beta.

49. Sea

$$ECM = E[(y-h(x))^2|X]$$

P.D.

$$min_f ECM(f) = E(Y|X=x)$$

$$\begin{aligned} min_{f_X} E[(y-h(x))^2|X] &= min_c E[(y-c)^2|X] \\ &= min_c[E(y^2|X) - 2E(yc|X=x) + E(c^2|X=x)] \\ \rightarrow -2E(Y|X) + 2c = 0 \end{aligned}$$

$$\rightarrow f_X(x) = E(Y|X)$$

50. Sean

$$u = \frac{y_1-\mu_1}{\sigma_1} \qquad v = \frac{y_2-\mu_2}{\sigma_2}$$

$$\begin{aligned} f(y_2|y_1) &= \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}exp[\frac{-1}{1-\rho^2}(u^2-2\rho uv+v^2)]}{\frac{1}{2\pi\sigma_1}e^{-u^2/2}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}exp[\frac{-1}{2}(\frac{v-\rho u}{\sqrt{1-\rho^2}})^2] \end{aligned}$$

$$\rightarrow f(y_2|y_1) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}exp[\frac{-1}{2}\frac{y_2-(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(y_1-\mu_1))}{\sigma_2\sqrt{1-\rho^2}}]$$

$$\rightarrow f(y_2|y_1) \sim N(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(y_1-\mu_1), (\sigma_2\sqrt{1-\rho^2})^2)$$

51. Sea

$$X \sim N(0,1)$$

$$\begin{aligned} \phi_X(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt}e^{-x^2/2}\mathtt{d}x \\ &= e^{-t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x-it)^2/2}\mathtt{d}x \\ &= e^{-t^2/2} \end{aligned}$$

En general, sea

$$W = \mu + \sigma x, W \sim N(\mu, \sigma^2)$$

$$\begin{aligned}
\phi_W(t) &= E(e^{i(\mu * \sigma x)t}) \\
&= e^{i\mu t} E(e^{it\sigma x}) \\
&= e^{i\mu t} \phi_X(\sigma t) \\
&= e^{i\mu t} e^{-\sigma^2 t^2/2}
\end{aligned}$$

52.

$$f_{p|n} = \frac{f_p f_{n|p}}{f_n}$$

$$\begin{aligned}
f_n &= \int_0^1 \binom{n+m}{n} p^n (1-p)^m \mathtt{d}p \\
&= \binom{n+m}{n} \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}
\end{aligned}$$

$$f_{p|n} = \frac{p^n (1-p)^m}{\frac{\Gamma(n+1)\Gamma(m+1}{\Gamma(n+m+2)}}$$

53.

$$f_{X,Y} = e^{-(x+y)} I_{[0,\infty)}(x) I_{[0,\infty)}(y)$$

Sea

$$\begin{aligned}
Z &= \frac{X}{Y} & \leftrightarrow X = WZ \\
W &= Y & \leftrightarrow Y = W
\end{aligned}$$

$$\mathbf{J} = \begin{pmatrix} Z & W \\ 1 & 0 \end{pmatrix}$$
$$\rightarrow |\mathbf{J}| = W$$

$$\begin{aligned}
f_{(W,Z)} &= w * f_{(X,Y)}(wz, w) \\
&= w * e^{-w(z+1)} I_{[0,\infty)}(w) I_{[0,\infty)}(z)
\end{aligned}$$

$$\begin{aligned}
f_Z &= \int_0^\infty w e^{-w(z+1)} dw I_{[0,\infty)}(z) \\
&= \frac{1}{z+1} I_{[0,\infty)}(z)
\end{aligned}$$

54. Sea

$$X \sim Po(\theta),$$
$$Y \sim Po(\lambda),$$
$$Z = X + Y \sim Po(\theta + \lambda).$$

p.d.

$$f_{X|Z} \sim Bin(Z, \frac{\theta}{\theta + \lambda})$$

$$\begin{aligned} f_{X,Z} &= f_{(X,Y)}(x, z - x) \\ &= \frac{\theta^x}{x!} e^{-(\theta)} \frac{\lambda^{z-x}}{(z-x)!} e^{-\lambda} I_{[0,1,2,..,z]}(x) I_{[0,1,2,..)}(y) \end{aligned}$$

$$f_Z = \frac{(\lambda + \theta)^z}{z!} e^{-(\lambda+\theta)} I_{[0,1,2,..)}(z)$$

$$\begin{aligned} \frac{f_{X,Z}}{f_Z} &= \frac{\frac{\theta^x}{x!} e^{-(\theta)} \frac{\lambda^{z-x}}{(z-x)!} e^{-\lambda} I_{(0,1,2,..,z)}(x) I_{(0,1,2,..)}(y)}{\frac{(\lambda+\theta)^z}{z!} e^{-(\lambda+\theta)} I_{(0,1,2,..)}(z)} \\ &= \frac{z!}{(x-z)!x!} \theta^x \lambda^{z-x} (\lambda + \theta)^{-z} I_{(0,1,2,..,z)}(x) \\ &= \binom{z}{x} (\frac{\theta}{\theta + \lambda})^x (\frac{\lambda}{\theta + \lambda})^{z-x} I_{[0,1,2,..,z)}(x) \end{aligned}$$

$$f_{(X|Z)} \sim Bin(z, \frac{\theta}{\theta + \lambda})$$

55. Sean X_1, X_2, X_3,.., X_n v.a.i.i.d

$$P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$$
$$N \sim Geo(\alpha)$$
$$P(N = n) = \alpha(1 - \alpha)^n I_{1,2,3,4,...}(n)$$

$$Y = \sum_{i=1}^{n} x_i \quad x_i \sim Geo(\alpha)$$

$$\begin{aligned} &\rightarrow Y \sim BinNeg(n, \alpha) \\ &\rightarrow f_Y = \binom{r + y - 1}{y} \alpha^r (1 - \alpha)^y I_{1,2,3,4,...}(y) \end{aligned}$$

56. Sea

(a)

$$f_S = k(s^2+1) \quad I_{(1,4)}(s)$$

(b)

$$f_T = ct \quad I_{(0,600)}(t)$$

(c)

$$1 = k\int_1^4 (s^2+1)\mathtt{d}s \qquad \to k = \frac{1}{24}$$

(d)

$$1 = c\int_0^6 00t\mathtt{d}t \qquad \to c = \frac{1}{180000}$$

(e)

$$f_X = \begin{cases} 100 & \text{c.p.}0.11112, \\ 200 & \text{c.p.}0.33336, \\ 300 & \text{c.p.}0.5556. \end{cases}$$

(f)

$$f_Y = \begin{cases} 1.5 & \text{c.p.}0.1388, \\ 2.5 & \text{c.p.}0.3055, \\ 3.5 & \text{c.p.}0.5555 \end{cases}$$

(g)

$$f_Y = \begin{cases} x=100 & y=1.5 & \text{c.p. } 0.022224, \\ x=100 & y=2.5 & \text{c.p. } 0.044448, \\ x=100 & y=3.5 & \text{c.p. } 0.044448, \\ x=200 & y=1.5 & \text{c.p. } 0.181963, \\ x=200 & y=2.5 & \text{c.p. } 0.060654, \\ x=200 & y=3.5 & \text{c.p. } 0.090981, \\ x=300 & y=1.5 & \text{c.p. } 0.06486, \\ x=300 & y=2.5 & \text{c.p. } 0.20039, \\ x=300 & y=3.5 & \text{c.p. } 0.42007 \end{cases}$$

57. Sean X_1, X_2, X_3, ...,X_n v.a.i.i.d.
$f_{X_i} = 2X_i \qquad I_{(0,1)}(x)$

$$W = Y_n - Y_1$$
$$Z = Y_n$$
$$\to |J| = 1$$

$$f_{W,Z} = 4n^2(1-(z-w)^2)^{n-1}(z-w)z^{2n-1} \quad I_{(0,1)}(w)I_{(w,1)}(z)$$

$$f_W = \int_w^1 4n^2(1-(z-w)^2)^{n-1}(z-w)z^{2n-1}\mathtt{d}z \quad I_{(0,1)}(x)$$

$$E(W) = \int_w^1 \int_w^1 4n^2(1-(z-w)^2)^{n-1}(z-w)z^{2n-1}\mathtt{d}z\mathtt{d}w$$

58. Sean

$$f_X = \frac{1}{4} \quad I_{(-2,-1,1,2)}(x)$$

Y sea

$$Y = X^2$$

(a)

X/Y	1	4
-2	0	0.25
-1	0.25	0
1	0.25	0
2	0	0.25

(b)

$$\begin{aligned} CovXY &= E(XY) - E(X)E(Y) \\ &= \sum_x \sum_y xyf(x,y) - \sum_x xf(x) \sum_y yf(y) \\ &= 0 \end{aligned}$$

(c)

$$\rho_{XY} = 0$$

(d) Pero no son independientes porque Y siempre va a depender de X

59. Sea

$$X \sim U(0,1)$$

Y

$$Y = log\frac{X}{1-X}$$

$$P(Y \leq log\frac{X}{1-X}) = P(X \leq \frac{e^y}{1+e^y})$$

$$\rightarrow f_Y = (1+e^y)e^y - e^{2y}(1+e^y)^{-2}I_{(-\infty,\infty)}(Y)$$