Inferences on the ratio of normal means and other related problems

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Summary

The problem of making inferences about the ratio of two parameters has been addressed, both from the frequentist and Bayesian perspectives, by many authors over the last fifty years. Most of this work is concerned with the ratio of two normal means. In this paper, we review the most relevant results regarding the Bayesian analysis of the nomal case as well as some related, more general, problems.

Key words and phrases: Bayesian Analysis; Ratio-type parameters; Fieller's formula; Ratio of Normal Means; Linear models; Reference Prior; Elliptic distributions.

1 Introduction

The problem of making inferences about a quantity defined as the ratio of two unknown parameters has been extensively discussed in the literature. In particular, the situation where a statistical analysis is required for the ratio of two normal means has been investigated for many years in the context of bioassay (see, for example, Bliss, 1935a, 1935b; Irwin, 1937; Fieller, 1944, 1954; Finney, 1947, 1965; Cox, 1985; Srivastava, 1986 and Kelly, 2000). The simplest version of this problem may be stated as follows.

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ be two independent random samples such that

$$\begin{array}{rcl} X_i & \sim & N(x|\mu_1, \sigma^2); \quad i = 1, \ldots, n \\ Y_j & \sim & N(y|\mu_2, \sigma^2); \quad j = 1, \ldots, m \end{array} \qquad (1)$$

where both, μ_1 and μ_2, are unknown. Provided that $\mu_2 \neq 0$, the aim of the analysis is to produce inferences about the parameter $\phi = \mu_1/\mu_2$ which describes the relative magnitude of the means and, for certain types of bioassay, can be interpreted as the relative potency of two different treatments.

In a similar way, there are other specific applications where inferences are more adequate if expressed in terms of a ratio of means or some more general form of a ratio-type parameter. This is the case, for example, of calibration and some linear models (cf. Lindley and El-Sayyad, 1968).

In this paper, we firstly describe the frequentist results obtained where the objective is to produce interval estimates for ϕ in (1). In Section 3, a Bayesian solution for this basic model is reviewed and Section 4 is devoted to some extensions of the Bayesian results to more general settings. Finally, some final remarks and future research topics are included in Section 5.

2 Frequentist results

From a frequentist point of view, pointwise estimation of ϕ is rather straightforward. In particular, maximum likelihood yields $\hat{\phi} = \bar{x}/\bar{y}$. On the other hand, if the objective is to produce interval estimates for this parameter, the answer is not so clear. The standard frequentist procedure is based on the celebrated Fieller's formula (1944, 1954) which, for this case (let us take $\sigma = 1$), is quite simple. Let us define an auxiliary random variable $U = \bar{X} - \phi\bar{Y}$. Then, we have that

$$h(\phi) = \frac{U^2}{(\frac{1}{n}+\frac{\phi^2}{m})} \tag{2}$$

follows a $\chi^2_{(1)}$ distribution. $h(\phi)$ is known as the Fieller's pivotal quantity and the corresponding $(1-\alpha)\times 100\%$ confidence region for ϕ is given by

$$\mathcal{A} = \{\phi \in I\!R \mid h(\phi) \leq \chi^2_{(1)(1-\alpha)}\}.$$

This confidence region can also be derived using very simple ideas of hypotheses testing. Let us consider the hypotheses

$$H_0 : \phi = \phi_0 \quad vs. \quad H_1 : \phi \neq \phi_0 \tag{3}$$

for model (1), reparametrized in terms of ϕ and μ_2. The generalized likelihood-ratio test leads to a rejection region for H_0, given by

$$\mathcal{C} = \{(\boldsymbol{X},\boldsymbol{Y}) \mid L_p(\phi_0)/c(\boldsymbol{X},\boldsymbol{Y}) \leq k\} \tag{4}$$

where $c(\boldsymbol{X},\boldsymbol{Y})$ involves only the observed data, and

$$L_p(\phi_0) = (2\pi)^{-(n+m)/2}\exp(-\frac{1}{2}h(\phi_0))$$

is the corresponding profile likelihood. Thus, given $(\boldsymbol{X},\boldsymbol{Y})$, the hypothesis $H_0 : \phi = \phi_0$ will be *accepted* if and only if, as a function of ϕ_0,

$$L_p(\phi_0) \geq k^* \tag{5}$$

or equivalently,

$$h(\phi_0) \leq c^* \tag{6}$$

for some constants k^* and c^*. As a consequence, Fieller's confidence region can be interpreted as the set of values ϕ_0 for which the hypothesis $H_0 : \phi = \phi_0$ is not rejected at the appropriate significance level. In particular, a test of size α leads to a confidence region of level $(1-\alpha)\times 100\%$ and then, $c^* = \chi^2_{(1)(1-\alpha)}$. At this point arises the most remarkable characteristic of this procedure which has lead to a number of severe criticisms: the $(1-\alpha)$ quantile of a chi-square distribution goes to infinity as $\alpha \to 0$, whereas $h(\phi)$ is a continuous, *bounded* function of ϕ, for any fixed set of data. In particular, $h(\phi) \to h^*$ as $\mid \phi \mid \to \infty$ (a typical example is displayed in Figure 1, for the case $n = 1, m = 1, x = 1, y = 1$).

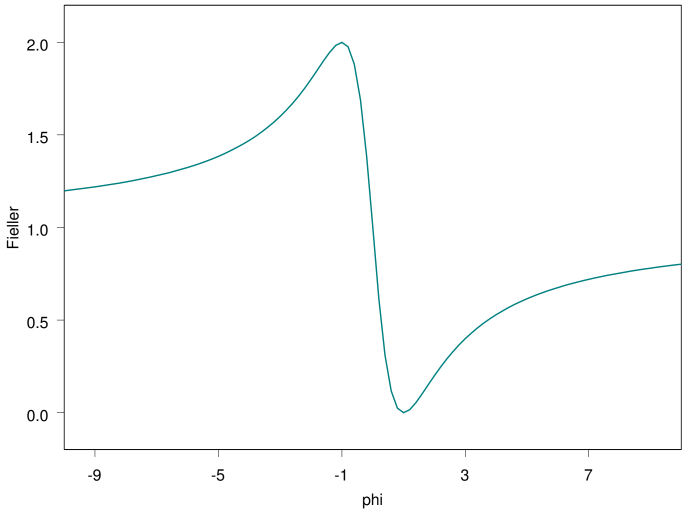

Figure 1: *Fieller's pivotal quantity* $h(\phi)$

As a consequence, we have that, for each sample $(\boldsymbol{X}, \boldsymbol{Y})$, it always exists a value α, strictly positive, such that (6) is true for every $\phi \in I\!R$. In other words, it can be proved that for every fixed set of data, the entire parameter space cannot reach a 100% confidence level. From a conceptual point of view, this result is inadmissible. Other minor problems are related to the fact that for some confidence levels, the region is defined as the union of two disjoint intervals of infinite lenght. In practice, these regions might be useless. In more general setting, Glesser and Hwang (1987) provide a detailed discussion of the difficulties that the frequentist approach faces when dealing with this class of problems.

It is interesting to notice that, despite these drawbacks, Fieller's formula and some generalizations of it are still widely used (see Raftery and Schweder, 1993, for an example). In any case, recalling the equivalent condition (5), it might be useful to review Fieller's main problem in terms of the (profile) likelihood. For this transformation, it happens that existence of a finite upper bound for $h(\phi)$ implies a strictly positive lower bound for $L_p(\phi)$ (see Figure 2). In particular, $L_p(\phi) \to m > 0$, as $\mid \phi \mid \to \infty$. This behaviour of the tails, implies that this function cannot longer be interpreted as a certain type of density for ϕ, as some non-Bayesian approaches suggest (see Sprott, 2000, for instance).

The idea of dealing with the likelihood function as a density for the parameter of interest is not intrinsically inadequate. The frequentist difficulty in this problem arises because of the existence of nuisance parameters (in this case, μ_2).

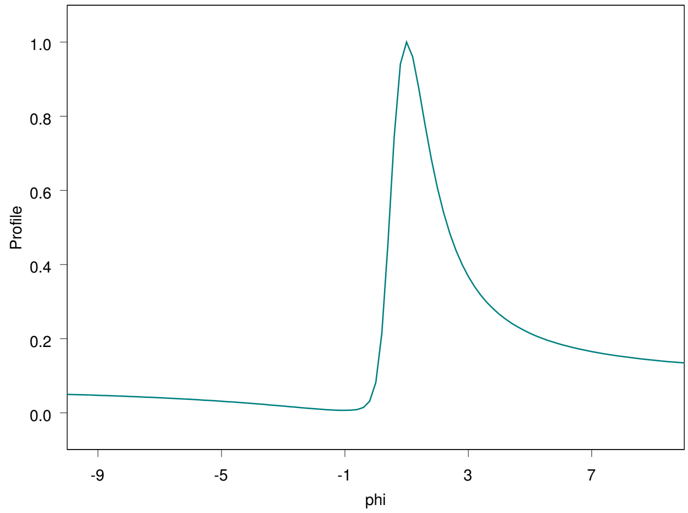

Figure 2: *Profile likelihood funtion* $L_p(\phi)$

Many authors have recognized that methods of inference applicable to the likelihood function which perform well in those cases where the parameter of interest is the only unknown quantity, might fail if, due to the presence of nuisance parameters, the *full* likelihood (in this case, $L(\phi, \mu_2)$), is replaced by some transformation (the profile likelihood $L_p(\phi)$, here).

In fact, a number of *corrections* have been proposed for the profile likelihood (Kalbfleish and Sprott, 1970; Cox and Hinkley, 1974; Barndorff-Nielsen, 1986; Cox and Ried, 1987; McCullagh and Tibshirani, 1990 are just some examples). It must be said however, that in general terms these corrections are not intended to produce a pseudo-density for the parameter of interest but to avoid other problems such as bias and inconsistency. Nevertheless, it is interesting to see how some of these corrections modify the tails of the profile likelihood for ϕ in this problem.

Let us consider, for example, the resulting *adjusted* profile likelihood $L_{ap}(\phi)$ obtained by Currie and Durbán (2000). These authors apply a procedure originally proposed by McCullagh and Tibshirani (1990) to produce a profile likelihood associated to an unbiased, and information unbiased, score function. The resulting modified likelihood is obtained as the product of the profile likelihood and a correction factor. It can be shown that

$$L_{ap}(\phi) \quad = \quad L_p(\phi) \times [1 + \tfrac{2 \log L_p(\phi)}{1+(X^2+Y^2)}]^{-1/2}$$

where again,

$$\lim_{|\phi|\to\infty} L_{ap}(\phi) \quad = \quad m^* > 0.$$

for some constant m^*. Therefore, and despite any other positive effects, it happens that this adjusted profile likelihood does not improves the tail behaviour of $L_p(\phi)$.

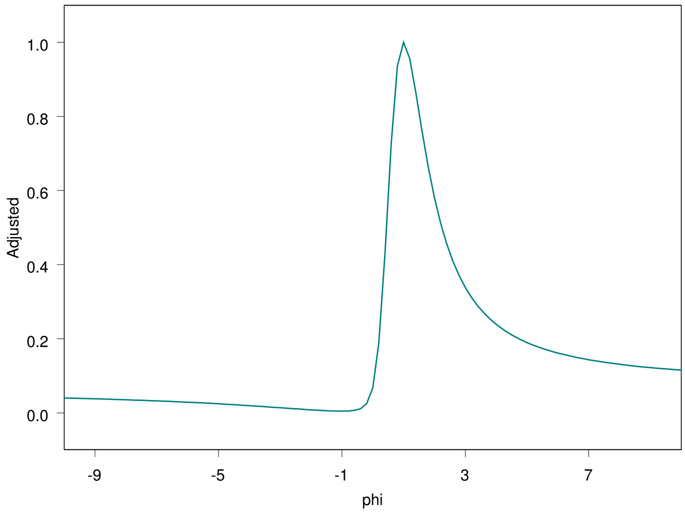

Figure 3: *Adjusted likelihood function* $L_{ap}(\phi)$

Another idea used to eliminate nuisance parameters is to integrate them out from the full likelihood. This might be considered as a *naive* Bayesian procedure involving a uniform prior distribution, even though it has been proposed by non-Bayesian authors (see Kalbfleish and Sprott, 1970). In any case, the integrated likelihood for the problem considered here, is given by

$$\begin{aligned} L_{UI}(\phi) &= \int L(\phi, \mu_2)\, d\mu_2 \\ &\propto L_p(\phi) \times (1+\phi^2)^{-1/2} \end{aligned} \tag{7}$$

Again, in this case the resulting function is obtained as the product of the profile likelihood and a correction factor. More interesting, this factor introduces a qualitative change for the tails of the new likelihood (see Figure 4). Now, we have that $L_{UI}(\phi) \to 0$, as $|\,\phi\,| \to \infty$. Unfortunately, these tails do not decay fast enough to get a probabilty model. In fact, it is clear from (7) that $L_{UI}(\phi)$ behaves as ϕ^{-1} when $|\,\phi\,| \to \infty$.

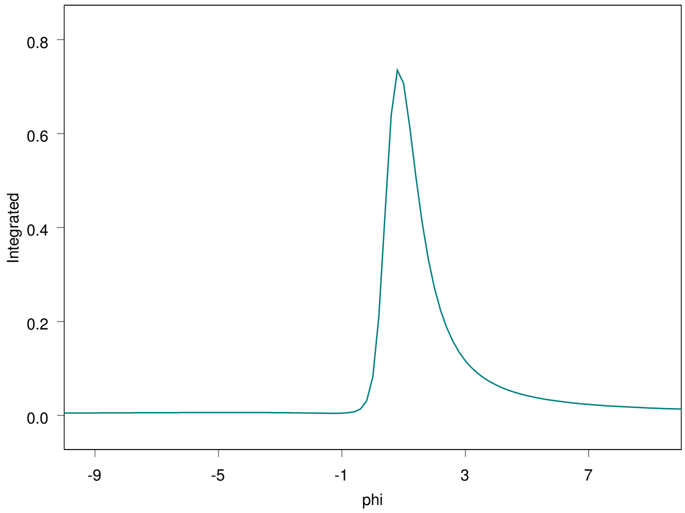

Figure 4: *Integrated likelihood function* $L_{UI}(\phi)$

A comprenhensive discussion of integrated likelihood methods as a tool for eliminating nuisance parameters is provided by Berger et al. (1999).

3 The Bayesian approach for the basic model

Form purely algorithmic point of view, Bayesian analysis of model (1) is another likelihood method. In fact, for the production of interval estimates, the full likelihood is replaced by *posterior* distribution of ϕ, which can be viewed as an integrated likelihood where the weight function is no longer uniform but proportional to the joint *prior* distribution for both parameters, ϕ and μ_2. Thus,

$$\begin{aligned} P(\phi \mid (\boldsymbol{X}, \boldsymbol{Y})) &\propto \int L(\phi, \mu_2)\, P(\phi, \mu_2)\, d\mu_2 \\ &\propto P(\phi) \times \int L(\phi, \mu_2)\, P(\mu_2 \mid \phi)\, d\mu_2. \end{aligned}$$

This posterior distribution is a proper probability model not only when the prior is proper but also for some cases where the initial information regarding the parameters is described through an improper function. In particular, if the prior conditional distribution for μ_2, given ϕ, is uniform we get

$$\begin{aligned} P(\phi \mid (\boldsymbol{X}, \boldsymbol{Y})) &\propto P(\phi) \times L_{UI}(\phi) \\ &\propto P(\phi) \times (1+\phi^2)^{-1/2} \times L_p(\phi) \end{aligned} \tag{8}$$

so that, the marginal prior for ϕ multiplied by $(1+\phi^2)^{-1/2}$ plays the role of a correction factor for the profile likelihood. Obviously, if the marginal prior for ϕ is proper, the resulting posterior is also proper. It must be noticed, however, that even under weaker conditions this marginal prior leads to a proper probabilty model. That is the case, for example, when $P(\phi)$ is improper but bounded with tails $O(\phi^{-1})$.

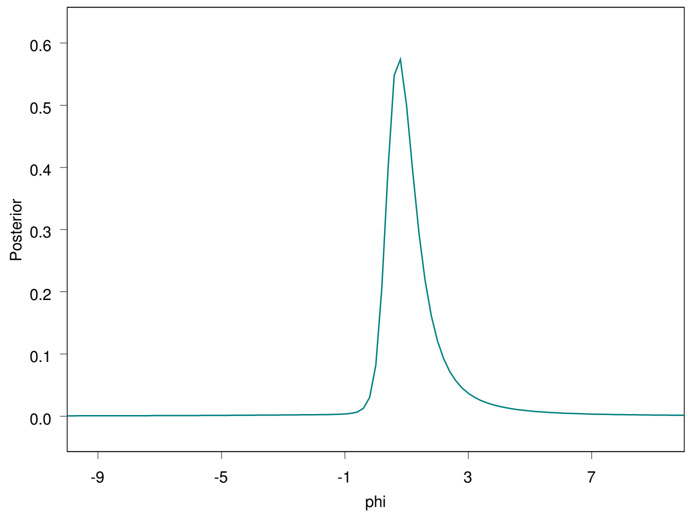

Figure 5: *Posterior distribution* $P(\phi \mid (\boldsymbol{X}, \boldsymbol{Y}))$

This result is particularly interesting because, following Bernardo (1979), the joint reference prior for ϕ and μ_2, when ϕ is the parameter of interest, is given by

$$\begin{aligned} \pi(\phi, \mu_2) &= \pi(\mu_2 \mid \phi) \times \pi(\phi) \\ &\propto 1 \times (1+\phi^2)^{-1/2} \end{aligned} \tag{9}$$

Therefore, even for the limiting case where the Bayesian approach assumes minimal prior information the resulting posterior, which can be interpreted as a transformed likelihood, results in a proper probabilty model for ϕ.

Model (1) can be generalized in several ways and some of these structures have already been analyzed from a Bayesian perspective. In particular, scale parameter σ can be unknown. This case has also been analized by Bernardo (1977), who derives the reference prior for (ϕ, μ_2, σ) when ϕ is again the parameter of interest. It must be noticed that a reference analysis when σ is unknown requires sample data enough to estimate this parameter.

Specifically, the reference prior when $n = m > 1$ in (1) and the ordered groups (cf. Berger and Bernardo, 1992) $\{\phi\}$ and $\{\mu_2, \sigma\}$ are considered, can be written as

$$\begin{aligned}\pi(\phi, \mu_2, \sigma) &= \pi(\mu_2, \sigma \mid \phi) \times \pi(\phi) \\ &\propto \pi(\mu_2 \mid \phi) \times \pi(\sigma \mid \phi) \times (1+\phi^2)^{-1/2} \\ &\propto 1 \times \sigma^{-2} \times (1+\phi^2)^{-1/2}\end{aligned}$$

so that, the corresponding posterior distribution for the parameter ϕ, applies the same correction factor $(1+\phi^2)^{-1/2}$ as (8) to the integrated likelihood

$$L_w(\phi) = \int L(\phi, \mu_2, \sigma)\ \sigma^{-2}\ d\mu_2\, d\sigma$$

If, alternatively, reference prior is derived when the relative importance of the parameters is described with the ordered groups $\{\phi\}$, $\{\mu_2\}$ and $\{\sigma\}$, the result is

$$\pi(\phi, \mu_2, \sigma) \propto 1 \times \sigma^{-1} \times (1+\phi^2)^{-1/2}$$

where only a change on the power of σ appears. Next section will review other, more general, models where reference priors are used to produce inferences for a ratio-type parameter. This analysis is particularly interesting if we accept that reference priors seem to define the minimum weight function to be used as a correction factor leading to proper probability models for these parameters.

4 Some generalizations

Let $\boldsymbol{Y} \in I\!R^n$ be a random vector following a multivariate normal distribution with mean vector $\boldsymbol{X}\theta$ and covariance matriz $\sigma^2\boldsymbol{I}$ where $\boldsymbol{X}$ is a fixed $(N \times k)$, full rank design matrix, $\theta \in I\!R^k (k < N)$ is a vector of unknown coefficients and $\sigma > 0$ is also unknown. For this linear regression setting, the parameter of interest is given by

$$\phi = \lambda_1^t\theta / \lambda_2^t\theta,$$

a ratio of two linear combinations of the regression coefficients. Here, λ_1 and λ_2 are assumed to be fixed, linearly independent vectors in $I\!R^k$ such that $\lambda_2^t\theta \neq 0$. Clearly, this structure includes, as a particular case, the ratio of normal means.

Other problems that can be accomodated as specific instances of this model are: slope ratio bioassay, parallel lines bioassay, switching regression regimes and calibration (see Mendoza, 1987; 1990 and Gosh et al., 1995, for example). Anyway, a Bayesian analysis for this model, has been provided by Mendoza (1988). There, results are obtained for a class of prior distributions that includes Bernardo's reference prior. First, the model is reparametrized in terms of $\beta = \boldsymbol{L}\,\theta$, where $\boldsymbol{L}$ is a full rank $k \times k$ matrix such that $\beta_i = \lambda_i\,\theta\,;\;\; 1 = 1, 2..$ Thus, we have that

$$\phi \;\; = \;\; \beta_1/\beta_2.$$

Then, reparametrizing the model in terms of $\phi, \beta_2, ..., \beta_k, \sigma^2$ and since $\beta_2, ..., \beta_k$ and σ^2 are nuisance parameters for this problem, the class of conditional priors

$$P(\beta_2, ..., \beta_k, \sigma \mid \phi) \;\; \propto \;\; \sigma^{-r},$$

where r is a positive constant, is proposed. This family includes different noninformative distributions. In addition, for the parameter of interest, a general prior $P(\phi)$ is considered. Therefore, an approximation to prior bieliefs regarding all parameters is given by

$$P(\phi, \beta_2, ..., \beta_k, \sigma) \;\; \propto \;\; P(\phi)\; \sigma^{-r}.$$

and the corresponding posterior is simply obtained as

$$\begin{aligned} P(\phi, \beta_2, ..., \beta_k, \sigma \mid \boldsymbol{Y}) \;\; &\propto \;\; P(\phi)\; \sigma^{-r}\; L(\phi, \beta_2, ..., \beta_k, \sigma) \\ &\propto \;\; P(\phi)\; \sigma^{-(N+r)} \exp\left[-(\boldsymbol{Y} - \boldsymbol{Z}\beta)^t(\boldsymbol{Y} - \boldsymbol{Z}\beta)/(2\sigma^2)\right] \end{aligned}$$

with $\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{L}^{-1}$ and $\beta^t = (\phi\beta_2, \beta_2, ..., \beta_k)$. Finally, integration provides the marginal posterior for ϕ

$$P(\phi \mid \boldsymbol{Y}) \;\; \propto \;\; P(\phi) \times L_w(\phi)$$

where the integrated likelihood is given by

$$L_w(\phi) \;\; = \;\; \{Q(\phi)\}^{-1/2}\,[\,\eta + h(\phi)\,]^{-(N+r-k)/2}$$

with $Q(\phi)$ a second degree polynomial with no real roots, $\eta = N - k$ and $h(\phi)$ the Fieller's pivotal quantity for ϕ in this problem, for which all drawbacks discused in Section 2 hold.

Again, since $\{Q(\phi)\}^{-1/2}$ has tails decaying to zero as ϕ^{-1}, it is clear that the posterior distribution for ϕ might or might not be a proper probability distribution depending on the prior. In particular, if $P(\phi)$ is proper or, at least, has tails $O(\phi)$, the posterior will be proper. Mendoza (1988) shows that the joint reference prior for this problem, when ϕ is the parameter of interest, and the ordered groups considered are $\{\phi\}$, $\{\beta_2, ..., \beta_k\}$ and $\{\sigma\}$, is given by

$$\pi(\phi, \beta_2, ..., \beta_k, \sigma) \quad \propto \quad \{Q(\phi)\}^{-1/2}\, \sigma^{-k} \tag{10}$$

so, that it belongs to the original class of priors with $P(\phi) = \{Q(\phi)\}^{-1/2}$ and $r = k$, generalizes Bernardo's results and leads to the posterior

$$P(\phi \mid \boldsymbol{Y}) \quad \propto \quad \{Q(\phi)\}^{-1}\, [\,\eta + h(\phi)\,]^{-N/2}$$

Therefore, the corresponding posterior is proper although it has no moments. In any case, not only the correction factor is adquate to get a proper posterior but it has the same structure as before, involving ϕ only through a second degree polynomial.

Another relevant generalization is due to Fernández and Steel (1998). These authors deal with the situation where a random sample, of size n, is obtained from a location-scale model

$$P(x \mid \mu_1, \sigma) \quad = \quad \sigma^{-1} f\{\sigma^{-1}(x - \mu_1)\},$$

where μ_1 is a location parameter, $\sigma > 0$ is scale parameter and $f(\cdot)$ is known density function. A second independent sample, of size m, is obtained from the model

$$P(x \mid \mu_2, \sigma) \quad = \quad \sigma^{-1} g\{\sigma^{-1}(x - \mu_2)\},$$

where μ_2 plays a role similar to that of μ_1 and $g(\cdot)$ is another density function. This structure is a generalization of model (1) that not only removes the normality assumption but does not require a common density for both samples nor existence of moments for the models. It is a remarkable general setting. For this case it is shown that, if the parameter of interest is $\phi = \mu_1/\mu_2$, the ratio of location parameters, and the ordered groups are defined as $\{\phi\}$, $\{\mu_2\}$ and $\{\sigma\}$, the joint reference prior is given by

$$\pi(\phi, \mu_2, \sigma) \quad \propto \quad \{Q(\phi)\}^{-1/2}\, \sigma^{-1}$$

where, as before, $Q(\phi)$ is a second degree polynomial. Moreover, if the corresponding information matrix is block-diagonal and $f(\cdot) = g(\cdot)$, then $Q(\phi) = (1 + \phi)^{-1/2}$ exactly.

More details concerning posterior distribution depend on the particular choice of $f(\cdot)$ and $g(\cdot)$. For example, the authors report that, for the case where both densities correspond to scale mixtures of normals, marginal posterior for ϕ is proper if an only if $n + m \geq 3$ and, as in previous cases, moments do not exist.

5 Concluding Remarks

The problem of making inferences about the ratio of two normal means is not only of practical importance but has also concentrated the attention of the statisticians for many years because of the difficulties associated with the use of the popular Fieller's pivotal quantity, when a frequentist analysis is conducted. These drawbacks are general and appear in many other problems where inferences for a ratio-type parameter are produced with this procedure. Fieller's controversial features are related, in a way, to the behaviour of the profile likelihood (and the uniform integrated likelihood) for this class of problems. From a Bayesian point of view, reference analysis provides a reasonable answer and it is interesting to notice that posterior densities can be directly expressed, in many cases, in terms of the profile or the integrated likelihood. Even more, prior densities act as a correction factor for the corresponding likelihood to get proper posterior densities. For the case of the ratio of normal means, this correction factor involves the parameter of interest only through a second order polynomial. This fact, however, is not particular for this model. Some generalizations have shown that, for example, this expression for the correction factor also appears if the parameter of interest is the ratio of two linear combinations of the coefficients in a normal linear model, a structure that includes many models where different parameters of this type might be analized. The same fact occurs for the very general case of the ratio of location parameters of two location-scale models. Even two different densities can be involved and the correction factor has the same structure.

The crucial condition here seems to be that of having the same variance or, at least, the same scale parameter. For the case of the ratio of normal means Mendoza and Gutiérrez-Peña (1999) have shown that if the variances are not equal, one gets a different result.

Other contribution which has explored other aspects of this problem is that of Barbieri et al. (2000) where Bayes factors are computed for testing the hypotheses $H_0 : \phi = \phi_0$ vs. $H_0 : \phi \neq \phi_0$ for model (1). They show that, in contrast with other *noninformative* priors, reference priors provide an adequate solution to some robustness and consistency problems.

In any case, some other extensions may be possible when populations to be compared have the same scale parameter. For example, this pattern could be studied for the family of elliptical distributions and elliptical linear models (Arellano-Valle et al., 2000) which are becoming increasingly popular.

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