BAUMGARTNER'S CONJECTURE AND BOUNDED FORCING AXIOMS

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ABSTRACT. We study the spectrum of forcing notions between the iterations of σ-closed followed by ccc forcings and the proper forcings. This includes the hierarchy of α-proper forcings for indecomposable countable ordinals α, the Axiom A forcings and forcings completely embeddable into an iteration of a σ-closed followed by a ccc forcing. For the latter class, we present an equivalent characterization in terms of Baumgartner's Axiom A. This resolves a conjecture of Baumgartner from the 1980s.

We also study the bounded forcing axioms for the hierarchy of α-proper forcings. Following ideas of Shelah we separate them for distinct countable indecomposable ordinals.

1. INTRODUCTION

After the discovery of finite support iteration [15] and Martin's Axiom [11], the technique of iterated forcing was dramatically extended through consideration of iterations with countable support. The classical paper of Baumgartner and Laver [4] on countable support iterations of Sacks forcing was developed further by Baumgartner into the theory of Axiom A forcing [3]. Baumgartner's Axiom A captures many of the common features of ccc, σ-closed and tree-like forcings and is sufficient to guarantee that ω_1 is not collapsed in a countable support iteration. The more general theory of proper forcing was later developed by Shelah [14] and has replaced Axiom A as the central notion in the theory of iterated forcing with countable support.

Together with the introduction of proper forcing, Shelah also considered the notion of α-proper forcing [14, Chapter V] for indecomposable countable ordinals α. Forcings which are α-proper for all countable ordinals are

called $<\omega_1$-proper. Ishiu [9] proved that the notions of Axiom A and $<\omega_1$-properness are in fact the same, meaning that, up to forcing-equivalence, they describe the same classes of quasi-orders. This also explained an earlier result of Koszmider [10] saying that Axiom A is preserved by countable-support iteration.

Baumgartner showed that the analogue of Martin's Axiom for proper forcing, called PFA (the Proper Forcing Axiom) is consistent relative to a supercompact cardinal and it is conjectured that its consistency strength is exactly that. PFA and the forcing axioms for the classes of α-proper forcings (written as PFA_α) were later systematically studied by Shelah [14]. However, a still very useful weakening of PFA considered by Goldstern and Shelah [7] and called BPFA (the Bounded Proper Forcing Axiom) turned out to have much lower consistency strength, below that of a Mahlo cardinal. In addition, some important consequences of PFA, such as the Todorčević–Veličković result [19, 5] that $c = \aleph_2$ holds under PFA, were shown to also follow from BPFA [13] (see also [6]). On the other hand, one should remember that the proof of Todorčević–Veličković in fact only uses $FA(\sigma\text{-closed} * ccc)$, i.e. the forcing axiom for the class of forcings completely embeddable into an iteration of σ-closed followed by ccc forcing. We will say that a forcing is *embeddable into σ-closed $*$ ccc* if it is forcing-equivalent to a forcing which can be completely embedded into an iteration $P * Q$ where P is σ-closed and Q is ccc.

Back in the 1980s, when Axiom A and proper forcing were invented, Baumgartner conjectured that every forcing satisfying Axiom A can be embedded into an iteration of a σ-closed followed by a ccc forcing. This would of course mean that the two classes are the same, up to forcing-equivalence. Probably, the first motivation came with the Mathias forcing and its decomposition into $P(\omega)/\text{fin}$ followed by the Mathias forcing with an ultrafilter. Later, the conjecture was confirmed for the Sacks forcing and other tree-like forcing notions in [8]. Miyamoto [12] proved it for the iterations of a ccc followed by a σ-closed forcing. Recently, Zapletal proved that in most cases if an idealized forcing is proper, then it is in fact embeddable into σ-closed $*$ ccc [22, Theorems 4.1.5, 4.2.4, 4.3.26, 4.5.9, Lemma 4.7.7].

We introduce the notion of a *strong Axiom A forcing* (for a precise definition see Section 2), which essentially says that the forcing notion satisfies Axiom A after taking a product with any σ-closed forcing. We prove the following characterization.

Theorem 1.1. *Let P be a forcing notion. The following are equivalent*

(i) *P satisfies strong Axiom A,*
(ii) *P is embeddable into σ-closed $*$ ccc.*

Theorem 1.1 is in the spirit of Baumgartner's conjecture as it shows that there is a close connection between Axiom A and embeddability into σ-closed $*$ ccc. This characterization, however, cannot be strengthened to the

one suggested by Baumgartner and Theorem 1.1 leads also to the following counterexample, which refutes the original conjecture.

Corollary 1.2. *There is an Axiom A forcing notion which is not embeddable into σ-closed $\ast$ ccc. It is of the form ccc $\ast$ σ-closed $\ast$ ccc.*

Given a class of forcing notions $\mathcal{S}$, the *Bounded Forcing Axiom for $\mathcal{S}$*, denoted by $\mathrm{BFA}(\mathcal{S})$, is the statement that for each complete Boolean algebra B in $\mathcal{S}$ and any collection $\mathcal{D}$ of ω_1-many size at most ω_1 predense subsets of B, there is a filter on B which intersects each element of $\mathcal{D}$. An equivalent form of of $\mathrm{BFA}(\mathcal{S})$, due independently to Bagaria [2] and Stavi–Väänänen [16], states that $H(\omega_2)^V$ is Σ_1-elementary in $H(\omega_2)^{V^B}$ for any complete Boolean algebra B in $\mathcal{S}$.

The Bounded Forcing Axiom for the class of ccc forcing notions is equivalent to Martin's Axiom and the Bounded Forcing Axiom for the class of proper forcings is exactly BPFA. In fact, there is a whole spectrum of bounded forcing axioms, namely the Bounded Forcing Axioms for the classes of α-proper forcing notions (written as BPFA_α), where α can be any countable indecomposable ordinal. There is also the Bounded Forcing Axiom for $<\omega_1$-proper forcing notions, which is (a priori) weaker than all BPFA_α. By the result of Ishiu, it is equivalent to the Bounded Forcing Axiom for the class of Axiom A forcings, also denoted by BAAFA. A still (a priori) weaker variation is the Bounded Forcing Axiom for the class of forcings embeddable into σ-closed $\ast$ ccc. We denote this axiom by $\mathrm{BFA}(\sigma$-closed $\ast$ ccc). Todorčević showed (see [18] or [1, Lemma 2.4]) that the consistency strength of $\mathrm{BFA}(\sigma$-closed $\ast$ ccc) is the same as of BPFA, i.e. a reflecting cardinal. This implies that actually all the axioms along this hierarchy have the same consistency strength.

In [20] Weinert showed that BAAFA is strictly weaker than BPFA, relative to a reflecting cardinal. In [14, Chapter XVII, pages 837–838] Shelah mentions that the forcing axioms PFA_α for indecomposable countable ordinals α can be separated by the full club guessing principles. Following this idea, we separate the axioms BPFA_α for indecomposable countable ordinals. Here, however, we consider a hierarchy of weak club guessing principles TWCG_α (for a definition see Section 3) and show the following

Theorem 1.3. *For indecomposable ordinals $\alpha < \beta < \omega_1$ the axiom BPFA_α (or PFA_α) is consistent with TWCG_α, relative to a reflecting cardinal (or a supercompact), whereas BPFA_α is inconsistent with TWCG_β.*

The weak club guessing principles were introduced already by Shelah, who considered them as a variant of the full (or tail) club guessing principles (cf. [9]). Theorem 1.3 refines the separation of the axioms PFA_α in terms of the full club guessing principles. We also show the following.

Theorem 1.4. *For indecomposable ordinals $\alpha < \beta < \omega_1$, the principle TWCG_β implies TWCG_α and TWCG_α does not imply TWCG_β.*

This paper is organized as follows. Section 2 contains the characterization of forcings embeddable into σ-closed $*$ ccc. Section 3 contains the results on the weak club guessing principles and the bounded forcing axioms for α-proper forcings.

1.1. **Remark.** After this work has been done, we have learnt that in the 1980s Todorčević also derived Corollary 1.2 with different methods from his results on the S-space problem in [17, Section 2]; that proof has, however, never been published.

2. Forcings embeddable into σ-closed $*$ ccc

Recall that a forcing notion P satisfies the *uniform Axiom A* if there is an ordering $\leq_0$ on P refining its original ordering such that any $\leq_0$-descending ω-sequence has a $\leq_0$-lower bound and for any antichain A in P any condition can be $\leq_0$-extended to become compatible with at most countably many elements of A. By a *quasi-order* we mean a reflexive and transitive relation.

Ishiu showed [9, Theorem 4.3] that, up to forcing-equivalence, Axiom A and uniform Axiom A are equivalent and describe precisely the class of $<\omega_1$-proper quasi-orders. More precisely, he showed that if P is an Axiom A forcing notion, then there is a quasi-order P' which is forcing-equivalent to P and an ordering $\leq_0$ on P' such that P' satisfies the uniform Axiom A via $\leq_0$. This is a motivation for the following definition.

Definition 2.1. A forcing notion P satisfies *strong Axiom A* if there a quasi-order P', forcing-equivalent to P, with an ordering $\leq_0$ on P' such that for any σ-closed forcing S the product $S \times P'$ satisfies uniform Axiom A via $\leq_S \times \leq_0$.

Any forcing of the form $R * \dot{Q}$, where R is σ-closed and $\dot{Q}$ is forced to be ccc, satisfies the uniform Axiom A. The ordering $\leq_0$ on $R * \dot{Q}$ is simply $\leq_R \times \dot{=}$, i.e. $(r_1, \dot{q}_1) \leq_0 (r_0, \dot{q}_0)$ if $r_1 \leq_R r_0$ and $r_1 \Vdash \dot{q}_0 = \dot{q}_1$. To see that $\leq_0$ witnesses the uniform Axiom A, take an antichain $\mathcal{A}$ in $R * \dot{Q}$ and a condition $(r_0, \dot{q}_0) \in R * \dot{Q}$. Pick any R-generic filter G over V through r_0 and note that in $V[G]$ we have that $\{(\dot{q})/G : \exists r \in G \ (r, \dot{q}/G) \in \mathcal{A}\}$ is an antichain in $\dot{Q}/G$ and hence it is countable by the assumption that $R \Vdash \dot{Q}$ is ccc. Note that for each $(\dot{q})/G$ in the above set, there is only one $r \in G$ such that $(r, \dot{q}) \in \mathcal{A}$, as $\mathcal{A}$ is an antichain. Since R does not add new countable subsets of the ground model, there is a countable $\mathcal{A}_0 \subseteq \mathcal{A}$ in V such that for some condition $r' \in G$ we have

$$(*) \qquad\qquad r' \Vdash \{(r, \dot{q}) \in \mathcal{A} : r \in \dot{G}\} = \mathcal{A}_0.$$

Enumerate $\mathcal{A}_0$ as $\{(r'_n, \dot{q}_n) : n < \omega\}$. Since r_0, r' and all the r'_n are in G, we can find $r_1 \in R$ extending all these conditions. Now we have that $(r_1, \dot{q}_0) \leq_0 (r_0, \dot{q}_0)$ and it is enough to check that $\{(r, \dot{q}) \in \mathcal{A} : (r, \dot{q})$ is compatible with $(r_1, \dot{q}_0)\}$ is contained in $\mathcal{A}_0$. But if $(r'', \dot{q}'') \in \mathcal{A} \setminus \mathcal{A}_0$ were

compatible with $(r_1, \dot{q}_0)$, then forcing with a filter G such that $r'', r_1 \in G$ would give that $(r'', \dot{q}'') \in \{(r, \dot{q}) \in \mathcal{A} : r \in G\}$, contradicting $(*)$.

Recall that if A is a complete Boolean algebra and B is a complete Boolean subalgebra of A, then the *projection* $\pi : A \to B$ is defined as follows: $\pi(a) = \bigwedge \{b \in B : a \leq b\}$, where the Boolean operation is computed in either of the two Boolean algebras.

Now we prove Theorem 1.1. The proof will use an old idea of Groszek [8].

Proof of Theorem 1.1. (ii)$\Rightarrow$(i). Suppose $P \lessdot R * \dot{Q}$, where R is σ-closed and $\dot{Q}$ is forced to be ccc. Without loss of generality assume that P is a complete Boolean subalgebra of $\mathrm{ro}(R * \dot{Q})$ and let $\pi : \mathrm{ro}(R * \dot{Q}) \to P$ be the projection. Let

$$P' = \{(p, (r, \dot{q})) : p \in P, (r, \dot{q}) \in R * \dot{Q} \text{ and } p \wedge (r, \dot{q}) \neq 0\},$$

where the Boolean operation is computed in $\mathrm{ro}(R * \dot{Q})$. Consider the function $\pi' : P' \to P$ defined as:

$$\pi'((p, (r, \dot{q}))) = p \wedge \pi((r, \dot{q}))$$

and define the order $\leq_{P'}$ on P' as follows: $(p_1, (r_1, \dot{q}_1)) \leq_{P'} (p_0, (r_0, \dot{q}_0))$ if $\pi'((p_1, (r_1, \dot{q}_1))) \leq_P \pi'((p_0, (r_0, \dot{q}_0)))$. Thus P' becomes a quasi-order with $\leq_{P'}$. Note that the definition of $\leq_{P'}$ implies that the function π' is a dense embedding from P' to P, hence P' and P are forcing-equivalent.

Recall that on $R * \dot{Q}$ we have the natural ordering $\leq_R \times \dot{=}$ (see remarks preceeding this theorem) to witness uniform Axiom A. Let $\leq_0$ on P' be defined as follows: $(p_1, (r_1, \dot{q}_1)) \leq_0 (p_0, (r_0, \dot{q}_0))$ if $p_1 = p_0$, $r_1 \leq_R r_0$ and $r_1 \Vdash \dot{q}_1 = \dot{q}_0$. Now we claim that this $\leq_0$ witnesses the strong Axiom A.

Let S be a σ-closed forcing notion. We need to check that $S \times P'$ satisfies uniform Axiom A via $\leq_S \times \leq_0$. It is clear that $S \times P'$ is σ-closed with respect to $\leq_S \times \leq_0$. Take an antichain A in $S \times P'$, $s \in S$ and $(p, (r, \dot{q})) \in P'$. Via $\mathrm{id} \times \pi'$ we get an antichain A' in $S \times P$. As every element of $\mathrm{ro}(R * \dot{Q})$ is a supremum of an antichain in $R * \dot{Q}$, we can refine the antichain A' to an antichain A'' such that

(a) every element of A'' is of the form $(s, (r, \dot{q}))$ for some $s \in S$ and $(r, \dot{q}) \in R * \dot{Q}$,

(b) every element of A' is the supremum of a subset of A''.

Now, A'' is an antichain in $S \times (R * \dot{Q})$. The latter is the same as $(S \times R) * \dot{Q}$ (where $\dot{Q}$, as an R-name naturally becomes an $S \times R$-name). We need the following lemma.

Lemma 2.2. *Let T be a σ-closed forcing notion and C be ccc. Then*

$$T \Vdash \check{C} \text{ is ccc.}$$

Proof. Suppose not. Let $\{\dot{c}_\alpha : \alpha < \omega_1\}$ be a T-name for an antichain in $\check{C}$. Since T is σ-closed, we can build a descending sequence $\langle t_\alpha \in T : \alpha < \omega_1 \rangle$ and a sequence of conditions $\langle c_\alpha \in C : \alpha < \omega_1 \rangle$ such that

$$t_\alpha \Vdash \dot{c}_\alpha = \check{c}_\alpha.$$

But then $\{c_\alpha : \alpha < \omega_1\}$ is an uncountable antichain in C, a contradiction.
$\square$

Now, Lemma 2.2 implies that if G is any R-generic over V, then in $V[G]$ we have
$$S \Vdash \dot{Q}/G \text{ is ccc.}$$
This means that $R \Vdash$ "$S \Vdash \dot{Q}$ is ccc", or in other words, $R \times S \Vdash \dot{Q}$ is ccc. Since $S \times R = R \times S$, by the remarks preceeding this theorem, we get that $\leq_{S \times R} \times \doteq$ witnesses uniform Axiom A for $(S \times R) * \dot{Q}$.

Therefore, there are $s' \leq_S s$ and $r' \leq_R r$ such that $(s', (r', \dot{q}))$ is compatible with only countably many elements of A''. By (b) above, $(s', r', \dot{q})$ is compatible with only countably many elements of A' and so is $(s', \pi(r', \dot{q}))$ since $A' \subseteq S \times P$. Since $(s', \pi'(p, (r', \dot{q}))) \leq_{S \times P} (s', \pi(r, \dot{q}))$ and by the definition of $\leq_{P'}$, we get that $(s', (p, (r', \dot{q})))$ is compatible with only countably many elements of A. We also have
$$(s', (p, (r', \dot{q}))) \leq_S \times \leq_0 (s, (p, (r, \dot{q}))),$$
hence $\leq_S \times \leq_0$ witnesses uniform Axiom A for $S \times P'$. This ends the proof of implication (ii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii). Suppose P satisfies strong Axiom A. Since embeddability into σ-closed $*$ ccc is invariant under forcing-equivalence, we can assume that the ordering $\leq_0$ witnessing strong Axiom A is defined on P. We shall construct a σ-closed forcing notion R and an R-name $\dot{Q}$ for a ccc forcing such that $P \lessdot R * \dot{Q}$. Let R be the forcing with countable subsets of P ordered as follows: for $\pi_0, \pi_1 \subseteq P$ countable write $\pi_1 \leq \pi_0$ if

- for each $p \in \pi_0$ there is $q \in \pi_1$ such that $q \leq_0 p$,
- for each $q \in \pi_1$ the set π_0 is predense below q.

Note that R is σ-closed. In any R-generic extension the union of the countable subsets of P which belong to the generic filter forms a suborder of P. Let $\dot{Q}$ be the canonical name for this subset. We will show that $P \lessdot R * \dot{Q}$ and that $\dot{Q}$ is forced to be ccc.

Lemma 2.3. *The forcing $R * \dot{Q}$ adds a generic filter for P.*

Proof. We show that R forces that the $\dot{Q}$-generic filter is P-generic over V. It is enough to show that for any dense open set $D \subseteq P$ and $p \in P$ the set
$$\{\pi \in R : (\pi \Vdash p \notin \dot{Q}) \vee (\exists d \in \pi \ p \in D \wedge d \leq p)\}$$
is dense in R. Take any $\pi \in R$ and suppose $\pi \Vdash p \in \dot{Q}$. There is $\pi' \leq \pi$ and $p' \leq p$ such that $p' \in \pi'$. Pick $d \in D$ such that $d \leq p'$. Then $\pi' \cup \{d\} \leq \pi$ is as needed.
$\square$

Note now that for any $\pi \in R$ we have

$(\star)$ $\qquad\qquad\qquad\qquad \pi \Vdash \pi$ is predense in $\dot{Q}$.

Indeed, if $\pi' \leq \pi$ and $\pi' \Vdash p \in \dot{Q}$, then there is $\pi'' \leq \pi'$ and $q \in \pi''$ such that $q \leq p$. Since $\pi'' \leq \pi$, there is $r \in \pi$ and $t \leq r, q$. Now $\pi'' \cup \{t\} \Vdash t \leq r, q$.

We will be done once we prove the following.

Lemma 2.4. *R forces that $\dot{Q}$ is ccc.*

Proof. Suppose that $\dot{A}$ is an R-name for an uncountable antichain in $\dot{Q}$. Assume that $\dot{A}$ is forced to be of cardinality ω_1, namely $R \Vdash \dot{A} = \{\dot{a}_\alpha : \alpha < \omega_1\}$.

Sublemma 2.5. *For each $\pi \in R$ and $p \in \pi$ there are $\pi' \leq \pi$, $p' \leq_0 p$ such that $p' \in \pi'$ and a countable $A_p \subseteq P$ such that*

$$\pi' \Vdash \{a \in \dot{A} : a \text{ is incompatible with } p'\} \subseteq A_p.$$

Proof. We build an antichain in $R \times P$. Let $C_0 \subseteq R$ be a maximal antichain below π deciding $\dot{a}_0$ and such that for every $\rho \in C_0$ there is $b^\rho \in \rho$ such that $b^\rho \leq a$, where $a \in P$ is such that $\rho \Vdash a = \dot{a}_0$. Let $D_0 = \{(\rho, b^\rho) : \rho \in C_0\}$. For $\xi < \omega_1$ use the fact that R is σ-closed to find a maximal antichain C_ξ below π which refines all C_α for $\alpha < \xi$, decides $\dot{a}_\xi$ and for every $\rho \in C_\xi$ there is $b^\rho \in \rho$ such that $b^\rho \leq a$, where $a \in P$ is such that $\rho \Vdash a = \dot{a}_\xi$. Let $D_\xi = \{(\rho, b^\rho) : \rho \in C_\xi\}$.

Now $D = \bigcup_{\xi < \omega_1} D_\xi$ is an antichain in $R \times P$. To see that it is enough to check that if $\xi_0 < \xi_1$, $(\rho_0, b^{\rho_0}) \in D_{\xi_0}$, $(\rho_1, b^{\rho_1}) \in D_{\xi_1}$ and $\rho_1 \leq \rho_0$, then b^{ρ_0} and b^{ρ_1} are incompatible in P. Suppose $c \leq b^{\rho_0}, b^{\rho_1}$ and put $\rho = \rho_1 \cup \{c\}$. Then

$$\rho \Vdash c \in \dot{Q} \text{ and } c \leq b^{\rho_1}, b^{\rho_0}$$

and hence $\rho \Vdash \dot{a}_{\xi_0}, \dot{a}_{\xi_1}$ are compatible. This is a contradiction.

Since $R \times P$ satisfies uniform Axiom A via $\leq \times \leq_0$, we can find $\sigma \leq \pi$, $p' \leq_0 p$ and a countable subset $D' \subseteq D$ such that

$$\{(\rho, a) \in D : (\rho, a) \text{ is incompatible with } (\sigma, p')\} \subseteq D'.$$

Let $A_p = \{a \in P : \exists \rho \in R \ (\rho, a) \in D'\}$. Put $\pi' = \sigma \cup \{p'\}$. □

Take now any $\pi \in R$. Using Sublemma 2.5 and a bookkepping argument we find a sequence $\langle \pi_n \in R : n < \omega \rangle$ such that $\pi_0 = \pi$ and for each $n < \omega$ and $p \in \pi_n$ there is $m_p > n$, $p' \in \pi_{m_p}$ such that $p' \leq_0 p$ and there is a countable $A_p \subseteq P$ such that

$(\star\star)$ $\qquad\qquad \pi_{m_p} \Vdash \{a \in \dot{A} : a \text{ is incompatible with } p'\} \subseteq A_p.$

For each $p \in \bigcup_{n < \omega} \pi_n$ construct a sequence $p_n \in P$ such that $p_0 = p' \in \pi_{m_p}$ and if $p_n \in \pi_m$, then $p_{n+1} \in \pi_{m+1}$ is such that $p_{n+1} \leq_0 p_n$. Let r_p be any condition such that $r_p \leq_0 p_n$ for all $n < \omega$..

We define π_ω as the family of all such r_p for $p \in \bigcup_{n < \omega} \pi_n$. Note that $\pi_\omega \leq \pi_n$ for each n and by $(\star)$ and $(\star\star)$ we have that

$$\pi_\omega \Vdash \bigcup\{A_p : p \in \bigcup_{n < \omega} \pi_n\} \text{ is predense in } \dot{Q}.$$

This contradicts the assumption that $\dot{A}$ is forced to be uncountable. □

This ends the proof of the implication (ii)$\Rightarrow$(i). □

Now we prove Corollary 1.2.

Proof of Corollary 1.2. Recall the example [14, Chapter XVII, Observation 2.12] of two proper forcing notions whose product collapses ω_1. The first of them is σ-closed and the other is an iteration of the form ccc $* \sigma$-closed $*$ ccc. Thus, the latter does not satisfy strong Axiom A but is forcing-equivalent to an Axiom A forcing, since it is $<\omega_1$-proper. It is not embeddable into σ-closed $*$ ccc by Theorem 1.1.   □

3. Bounded forcing axioms and weak club guessing

Definition 3.1. Let $\kappa > \omega$ be a regular cardinal, α an ordinal and $\mathcal{M} = \{M_\varepsilon : \varepsilon \in \alpha\}$ be a sequence of countable elementary substructures of $H(\kappa)$. We say that $\mathcal{M}$ is an *internally approachable tower* if the following hold:

(i) $\{M_\varepsilon : \varepsilon \leq \eta\} \in M_{\eta+1}$ for every $\eta \in \alpha$ with $\eta + 1 \in \alpha$,
(ii) $M_\eta = \bigcup\{M_\varepsilon : \varepsilon < \eta\}$ for every limit ordinal $\eta \in \alpha$.

As usual, $H(\kappa)$ is the collection of all sets of hereditary cardinality less than κ. We will identify $H(\kappa)$ with the structure $\langle H(\kappa), \in, \lhd \rangle$, where $\lhd$ is a fixed well order of $H(\kappa)$.

Definition 3.2. Let P be a partial order and α a countable ordinal.

(a) Given $q \in P$ and $\mathcal{M} = \{M_\varepsilon : \varepsilon \in \alpha\}$ an internally approachable tower of countable elementary substructures of $H(\kappa)$ with $P \in M_0$, we say that q is *generic over* $\mathcal{M}$ if q forces that $\dot{G} \cap M_\varepsilon$ is generic over M_ε for every $\varepsilon \in \alpha$.
(b) P is α-*proper* if for every sufficiently large regular cardinal κ, for every internally approachable tower $\mathcal{M} = \{M_\varepsilon : \varepsilon \in \alpha\}$ as above and for every condition $p \in P \cap M_0$, there exists $q \leq p$ such that q is $(\mathcal{M}, P)$-generic. P is $<\omega_1$-proper if it is α-proper for each $\alpha < \omega_1$.

Note that if P is proper (i.e., 1-proper), then P is n-proper for every natural number n. Recall that a countable ordinal β is said to be *indecomposable* if there exists a nonzero ordinal τ such that $\beta = \omega^\tau$ (this is ordinal exponentiation). Equivalently, β is indecomposable if for every $\gamma < \beta$, the order type of the interval (γ, β) is equal to β. Now, if P is α-proper and β is the first indecomposable ordinal above α, then P is γ-proper for every $\gamma < \beta$.

Let α be an indecomposable ordinal. We denote by PFA_α the forcing axiom for the class of α-proper forcing notions. By BPFA_α we denote the bounded forcing axiom for this class.

Definition 3.3. An α-*ladder system* is a sequence $\bar{A} = \langle A_\beta : \beta < \omega_1 \rangle$ such that for each $\beta < \omega_1$, with α dividing β, the set A_β is a closed unbounded subset of β and $\text{ot}(A_\beta) = \alpha$. We will always assume that $\langle A_\beta(\tau) : \tau < \alpha \rangle$ is the increasing enumeration of the elements of A_β. We say that an α-ladder system $\langle A_\beta : \beta < \omega_1 \rangle$ is *thin* if for any $\beta < \omega_1$ the set $\{A_\gamma \cap \beta : \gamma \in \omega_1\}$ is countable.

Definition 3.4. The α-*Weak Club Guessing* principle, denoted by WCG_α says that there is an α-ladder system $\bar{A}$ such that for every club $D \subseteq \omega_1$ there is $\beta \in D$ such that α divides β and $\text{ot}(A_\beta \cap D) = \alpha$. The α-*Thin Weak Club Guessing* principle, denoted by TWCG_α, also asserts the existence of such an $\bar{A}$ but with the additional requirement of being thin.

Thin (full) club guessing ladder systems have been considered in the literature in [21, 9]. Zapletal mentions [21, Section 1.A] that their existence can be derived from $\diamondsuit$ and shows [21, Section 2] how to force one with a σ-closed forcing notion.

Theorem 3.5. *For indecomposable ordinals* $\alpha < \beta < \omega_1$, BPFA_α *implies the negation of* TWCG_β.

Proof. By the $\Sigma_1(H(\omega_2))$ generic absoluteness characterization of BPFA_α, it suffices to prove that for any thin β-club guessing sequence $\bar{A}$ there is an α-proper forcing notion shooting a club in ω_1 which is not guessed by $\bar{A}$.

Fix a thin β-club guessing sequence $\bar{A} = \langle A_\gamma : \gamma < \omega_1 \rangle$. We may assume $A_\gamma = \emptyset$ if β does not divide γ. Let P be the following forcing notion. Conditions in P are countable subsets C of ω_1 such that

- C is closed in the order topology,
- $\text{ot}(C \cap A_\gamma) < \beta$ for each $\gamma < \omega_1$ with β dividing γ.

The ordering $\leq_P$ on P is by end-extension. We need to show that P is α-proper. Let κ be a sufficiently large regular cardinal and let $\lhd$ be a well-ordering on $H(\kappa)$. Pick an internally approachable tower $\mathcal{M} = \langle M_\gamma : \gamma < \alpha \rangle$ of countable elementary submodels of $\langle H(\kappa), \in, \lhd \rangle$ such that $\bar{A} \in M_0$. Put $\rho_\gamma = M_\gamma \cap \omega_1$. Let $p \in M_0$ be any condition in P. We need to find a condition extending p and generic for the whole tower. For so doing, consider the $\lhd$-least ω-ladder system $\bar{B}$ and note that $\bar{B} \in M_0$.

Say that $X \subseteq \omega_1$ is $\mathcal{M}$-*accessible* if the order type of X is strictly less than ρ_0 and $X \cap \rho_\gamma \in M_{\gamma+1}$ for every $\gamma < \alpha$. Note that each A_γ is $\mathcal{M}$-accessible by thinness. For each $\mathcal{M}$-accessible $X \subseteq \omega_1$ we construct by induction a decreasing sequence of conditions $p(\gamma, X)$ for $\gamma \leq \alpha$ such that for each $\gamma \leq \alpha$ we have

  (i) $p(0, X) = p$,
  (ii) $p(\gamma, X)$ is a P-generic condition for $\langle M_\delta : \delta < \gamma \rangle$,
  (iii) if $\gamma = \delta + 1$, then $p(\gamma, X) \cap (A_{\rho_\delta} \cup X) \subseteq p \cup \{\rho_\varepsilon : \varepsilon \leq \delta\}$.
  (iv) $p(\gamma, X) \in M_\gamma$ for successor γ and $p(\gamma, X) \in M_{\gamma+1}$ for limit γ.

Here $M_{\alpha+1} = H(\kappa)$. In order to guarantee that (iv) holds, we will also require the following conditions:

  (v) $p(\gamma, X) = \bigcup_{n<\omega} p(B_\gamma(n), X \cup A_{\rho_\gamma})$ for limit γ,
  (vi) $p(\gamma + 1, X) = p(\gamma, X)$ for limit γ,
  (vii) if $\gamma < \alpha$ is zero or successor, then $p(\gamma+1, X)$ is the $\lhd$-least condition which extends $p(\gamma, X)$ and satisfies (i), (ii) and (iii).

Put $p(0, X) = p$. Suppose $\gamma \leq \alpha$ and $p(\delta, X)$ have been constructed for all $\delta < \gamma$. If γ is limit, then $p(\gamma, X)$ is defined as in (v). If $\gamma = \delta + 1$ and

δ is a limit, then $p(\gamma, X) = p(\delta, X)$. Suppose $\gamma = \delta + 1$ and δ is zero or a successor, in which case $p(\delta, X) \in M_\delta$. We need to show that there exists a condition extending $p(\delta, X)$ and satisfying (ii) and (iii).

Enumerate all dense open subsets of P in M_δ into a sequence $\langle D_n : n < \omega \rangle$ (assume $D_0 = P$) and inductively construct a decreasing sequence of conditions $p^n \in M_\delta \cap D_n$ such that $p^0 = p(\delta, X)$ and $p^n \cap (A_{\rho_\delta} \cup X) = p(\delta, X)$. Suppose $p^n \in M_\delta$ has been constructed and let $\eta^n = \sup(p_n)$. Consider the function $f : \omega_1 \setminus \eta^n \to \omega_1$ defined as follows: for $\nu \in \omega_1 \setminus \eta^n$ let q^ν be the $\lhd$-smallest condition which extends $p^n \cup \{\nu\}$ and belongs to D_{n+1}. Then we define $f(\nu)$ as the maximum of q^ν. Now let $E \subseteq \omega_1$ be the club of those points greater than η^n which are closed under f. Note that f and E are in M_δ, since they are definable from parameters in this model. It follows that

$$\text{ot}(E \cap \rho_\delta) = \rho_\delta > \text{ot}(A_{\rho_\delta} \cup (X \cap \rho_\delta)).$$

Choose two elements $\nu_0 < \nu_1 < \rho_\delta$ of E such that $[\nu_0, \nu_1] \cap (A_{\rho_\delta} \cup (X \cap \rho_\delta)) = \emptyset$. We can choose p^{n+1} to be q^{ν_0}.

Now the condition $\bigcup_{n<\omega} p^n \cup \{\rho_\delta\}$ is P-generic for M_δ and for the whole subtower $\langle M_\varepsilon : \varepsilon < \gamma \rangle$ and satisfies (ii) and (iii). Let $p(\gamma, X)$ be the $\lhd$-smallest condition with these properties and note that $p(\gamma, X) \in M_\gamma$, since this condition is definable (using the order $\lhd$ of $H(\kappa)$) from p, $X \cap \rho_\delta$ and $\langle M_\varepsilon : \varepsilon < \gamma \rangle$. This ends the successor step of the inductive construction. It is immediate that the condition $p(\alpha, \emptyset)$ is generic for the whole tower $\langle M_\gamma : \gamma < \alpha \rangle$. $\square$

The following proposition (due to Shelah) appears in [9, Proposition 3.5] for full club guessing ladder systems. The proof for weak club guessing is exactly the same. We provide it for the reader's convenience.

Proposition 3.6. *Let $\bar{A} = \langle A_\gamma : \gamma < \omega_1 \rangle$ be a thin β-ladder system and P a β-proper notion of forcing. If $\bar{A}$ witness TWCG_β, then $\bar{A}$ witnesses TWCG_β in any generic extension with P.*

Proof. Let $\dot{E}$ be a P-name for a club and p a condition in P. It suffices to prove that there exist an ordinal ρ^* and a condition $q \leq p$ such that q forces that the intersection of $\dot{E}$ with A_{ρ^*} has order type equal to β. For so doing, let κ be a sufficiently large regular cardinal and consider an internally approachable tower $\mathcal{M} = \langle M_\varepsilon : \varepsilon \in \omega_1 \rangle$ of countable elementary substructures of $H(\kappa)$ such that $\bar{A}$, P, $\dot{E}$ and p are in M_0. Let F be the club of those countable ordinals ρ such that $\rho = M_\rho \cap \omega_1$. Now, by TWCG_β (applied in V), there exist $\rho^* \in F$ such that $\text{ot}(A_{\rho^*} \cap F) = \beta$. Note that for each $\rho \in A_{\rho^*} \cap F$, any (M_ρ, P)-generic condition forces that $\rho \in \dot{E}$. So, it suffices to prove that there is a condition extending p which is generic for all elements of the tower $\mathcal{M}^* = \langle M_\varepsilon : \varepsilon \in A_{\rho^*} \cap F \rangle$. Given that P is β-proper, this can be reduced to proving that $\mathcal{M}^*$ is internally approachable, which is true by the assumptions that $\bar{A}$ is thin and $\mathcal{M}$ is internally approachable. $\square$

Corollary 3.7. *For every indecomposable ordinal $\gamma < \omega_1$ the principle* TWCG_γ *is consistent with* BPFA_γ *(or* PFA_γ*), relative to a reflecting cardinal (or a supercompact).*

Proof. We prove only the PFA version. The proof is very similar to the usual proof of the consistency of PFA, and so we omit the details. We start with a ground model with a supercompact satisfying TWCG_γ (there is one by the results of [21]). The generic extension that we need is obtained by a countable support iteration of length a supercompact cardinal, where in each step of the iteration we only consider names for γ-proper partial orders. Since the countable support iteration of γ-proper forcing notions is γ-proper [14, Chapter V, Theorem 3.5] and γ-proper forcing preserves TWCG_γ, we get a model of both, PFA_γ and TWCG_γ. $\square$

Together, Theorem 3.5 and Corollary 3.7 prove Theorem 1.3. The separation of the axioms PFA_α for indecomposable ordinals $\alpha < \omega_1$ appears in Shelah's [14, Chapter XVII, Remark 3.15]. We are not aware, however, if the separation with the bounded versions has ever appeared in the literature, so we mention it in the following corollary.

Corollary 3.8. *For indecomposable ordinals $\alpha < \beta < \omega_1$, BPFA_β (or PFA_β) does not imply BPFA_α, relative to a reflecting cardinal (or a supercompact).*

Proof. By Corollary 3.7 there is a model of BPFA_β (or PFA_β) and TWCG_β, relative to a reflecting cardinal (or a supercompact). It cannot satisfy BPFA_α by Theorem 3.5. $\square$

In the remaining part of this section we will prove Theorem 1.4. We will need an additional piece of notation. Given an indecomposable ordinal β and a cardinal $\kappa \leq \omega_1$, a (β, κ)-*system* is a sequence $\bar{A} = \langle A_\delta^\alpha : \alpha \in \kappa, \delta \in \omega_1 \rangle$ such that for every α and δ, with β dividing δ, the set A_δ^α is a closed unbounded subset of δ of order type β. A (β, κ)-system $\bar{A}$ is *thin* if for any $\gamma \in \omega_1$, the set $\{A_\delta^\alpha \cap \gamma : \alpha < \kappa, \delta \in \omega_1\}$ is countable.

Note that a (β, κ)-system $\bar{A}$ can be enumerated as $(\bar{A}_\delta : \delta < \omega_1)$, but then we must remember that $\bar{A}_\delta$ need not be cofinal in δ. Such enumerations will be used in the proof of Theorem 1.4 below.

The principle $\mathrm{WCG}_\beta^\kappa$ asserts the existence of a (β, κ)-system $\bar{A} = \langle A_\delta^\alpha : \alpha \in \kappa, \delta \in \omega_1 \rangle$ such that for every club $D \subseteq \omega_1$, there exists $\delta \in D$ and $\alpha \in \kappa$ such that β divides δ and $ot(A_\delta^\alpha \cap D) = \beta$. The principle $\mathrm{TWCG}_\beta^\kappa$ says exactly the same that $\mathrm{WCG}_\beta^\kappa$ with the additional requirement that $\bar{A}$ must be thin.

Lemma 3.9. *For any indecomposable ordinal β, TWCG_β is equivalent to $\mathrm{TWCG}_\beta^{\aleph_0}$ and the same holds for the non-thin versions.*

Proof. The two statements have the same proof. We only focus on the thin versions and we show that $\mathrm{TWCG}_\beta^{\aleph_0}$ implies TWCG_β. So, let $\langle A_\delta^n : n \in$

$\omega, \delta \in \omega_1\rangle$ be a $(\beta, \aleph_0)$-system witnessing $\mathrm{TWCG}_\beta^{\aleph_0}$. We define a thin β-ladder system $\langle B_\delta : \delta \in \omega_1\rangle$ as follows. First, for each δ divisible by β fix a cofinal sequence $\langle \delta_n : n \in \omega\rangle \subseteq \delta$ of order type ω. Define $B_\delta = \bigcup\{B_\delta^n : n \in \omega\}$, where B_δ^n is equal to $A_\delta^n \setminus \delta_n$. Now $\langle B_\delta : \delta \in \omega_1\rangle$ is a thin system. To see this, notice that for each $\gamma \in \omega_1$ if $\delta > \gamma$, $\delta \in \omega_1$ is divisible by β, then only finitely many of the δ_n are below γ and hence $B_\delta \cap \gamma$ is a union of finitely many of the sets $A_\delta^n \cap \gamma \setminus \delta_n$. The fact that $\langle B_\delta : \delta \in \omega_1\rangle$ witnesses TWCG_β follows directly from the assumption that $\langle A_\delta^n : n \in \omega, \delta \in \omega_1\rangle$ witnesses $\mathrm{TWCG}_\beta^{\aleph_0}$. $\qquad\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. The fact that TWCG_α does not imply TWCG_β follows directly from Theorem 1.3. Alternately, to derive this just in ZFC, one can start with a model of $\mathrm{TWCG}_\alpha + \mathrm{CH} + 2^{\aleph_1} = \aleph_2$ and then force with a countable-support iteration of length ω_2 of α-proper forcings, killing all β-thin club sequences.

Now we prove that TWCG_β implies TWCG_α. Assume TWCG_α fails. We will show by induction on $\alpha' \in [\alpha, \beta]$ that $\mathrm{TWCG}_{\alpha'}$ fails. In fact, we will show that if $\bar{A} = \{\bar{A}_\delta : \delta \in \omega_1\}$ enumerates a thin $(\alpha', \aleph_0)$-system, then there exists a club D such that for every $\delta \in \omega_1$ the intersection of $\bar{A}_\delta$ with D has order type strictly less than α. The case $\alpha' = \alpha$ follows from Lemma 3.9. Assume $\alpha' > \alpha$ and fix an enumeration $\{\bar{A}_\delta : \delta \in \omega_1\}$ of a thin $(\alpha', \aleph_0)$-system $\bar{A}$. For each ordinal δ find an increasing cofinal sequence $\{\delta_n : n \in \omega\} \subseteq \bar{A}_\delta$ of limit points of the set $\bar{A}_\delta$ such that the order types of

$$A'(\delta, 0) = \bar{A}_\delta \cap \delta_0$$

and

$$A'(\delta, n+1) = (\bar{A}_\delta \cap \delta_{n+1}) \setminus \delta_n$$

are indecomposable ordinals greater than or equal to α. Now, consider the thin system enumeration

$$A' = \{A'(\delta, n) : \delta \in \omega_1, n \in \omega\},$$

and note that for each indecomposable π in the semi-open interval $[\alpha, \alpha')$, the inductive hypothesis ensures the existence of a club C_π such that for every δ and for every n if the order type of $A'(\delta, n)$ is equal to π, then $\mathrm{ot}(A'(\delta, n) \cap C_\pi) < \alpha$. Note that if $\bar{A}$ is thin, then the set of those elements of A' whose order type is equal to π is a $(\pi, \aleph_0)$-system. Let C be the intersection of all the C_π. Now define the set $\bar{B}_\delta$ as follows:

$$\bar{B}_\delta = \{\delta_n : n \in \omega\} \cup \bigcup\{A'(\delta, n) \cap C : n \in \omega\}.$$

Note that this set has order type at most α. Also note that if $\gamma < \sup \bar{A}_\delta$, then $\bar{B}_\delta \cap \gamma$ is equal to the union of $\bar{A}_\delta \cap \gamma \cap C$ together with a finite subset of $\sup \bar{A}_\delta$. Therefore the system $\bar{B} = \{\bar{B}_\delta : \delta \in \omega_1\}$ is thin. Finally, find a club $D \subseteq C$ witnessing that the system $\bar{B}$ does not guess in the $(\alpha, \aleph_0)$-sense. Now D is as desired. $\qquad\square$

4. Remaining questions

There are some questions which this papers leaves open.

Question 4.1. *Is* $\mathrm{BFA}(\sigma\text{-closed} * \mathrm{ccc})$ *equivalent to* BAAFA?

Question 4.2. *Is strong Axiom A equivalent to the fact that the product with every σ-closed forcing is $<\omega_1$-proper?*

Question 4.3. *Does Theorem 1.4 hold for* WCG_α *in place of* TWCG_α?

Question 4.4. *Is every Axiom A forcing embeddable into a finite iteration of σ-closed and ccc forcings?*

Question 4.5. *Does* $\mathrm{BFA}(\sigma\text{-closed} * \mathrm{ccc})$ *imply* $2^{\aleph_0} = \aleph_2$?

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