COHERENT ADEQUATE FORCING AND PRESERVING CH

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ABSTRACT. We develop a general framework for forcing with coherent adequate sets on $H(\lambda)$ as side conditions, where $\lambda \geq \omega_2$ is a cardinal of uncountable cofinality. We describe a class of forcing posets which we call coherent adequate type forcings. The main theorem of the paper is that any coherent adequate type forcing preserves CH. We show that there exists a forcing poset for adding a club subset of ω_2 with finite conditions while preserving CH, solving a problem of Friedman [3].

The method of side conditions, invented by Todorčević ([10]), describes a style of forcing in which elementary substructures are included in the conditions of a forcing poset to ensure that the forcing poset preserves cardinals. Friedman ([3]) and Mitchell ([7]) independently took the first steps in generalizing the method from adding generic objects of size ω_1 to adding larger objects by defining forcing posets with finite conditions for adding a club subset of ω_2. Neeman ([9]) was the first to simplify the side conditions of Friedman and Mitchell and present a generally applicable technique for forcing on ω_2 with finite conditions.

Krueger ([4]) developed an alternative framework for forcing objects of size ω_2 with finite conditions, using adequate sets of models as side conditions. An adequate set of models consists of countable models which are pairwise membership comparable up to some initial segment. Later Krueger ([5]) introduced the idea of coherent adequate sets, which requires the existence of isomorphisms between certain models in an adequate set. This idea combined adequate sets with an isomorphism structure originally used by Todorčević [10] in the context of forcing on ω_1. Coherent adequate sets were applied in [5] to define a strongly proper forcing poset which forces $\square_{\omega_1}$.

The present paper makes advances on the framework of coherent adequate sets. We present a more general development in the context of $H(\lambda)$ for a cardinal $\lambda \geq \omega_2$ of uncountable cofinality, rather than just $H(\omega_2)$ as was treated in [5]. We define a class of forcing posets which we call coherent adequate type forcings. The main theorem of the paper is that any coherent adequate type forcing preserves CH. More generally, any coherent adequate type forcing on $H(\lambda)$, where $2^\omega < \lambda$ is a cardinal of uncountable cofinality, collapses 2^ω to have size ω_1, preserves $(2^\omega)^+$, and forces CH. We describe coherent adequate type forcings for adding a square sequence and for adding a club to a fat stationary subset of ω_2.

The forcing posets of Friedman, Mitchell, and Neeman for adding a club subset of ω_2 with finite conditions all force that $2^\omega = \omega_2$. Any forcing poset which has strongly generic conditions for countable models will add reals, including those

defined in this paper. These earlier forcings for adding clubs with finite conditions can be factored in many ways so that the quotient forcing also has strongly generic conditions in the intermediate extension. For this reason, these posets add ω_2 many distinct reals. Friedman ([3]) asked whether it is possible to add a club subset of ω_2 with finite conditions while preserving CH. We solve this problem by defining a forcing poset which adds a club to a fat stationary set and falls in the class of coherent adequate type forcings.

Finally we show that, under CH, the forcing poset consisting of finite coherent adequate subsets of $H(\lambda)$ ordered by inclusion, where $\lambda \geq \omega_2$ is regular, preserves ω_1 and is ω_2-c.c., and therefore preserves all cardinals.

Section 1 develops the basic ideas of adequate and coherent adequate sets. This development is self contained. Differences between the current paper and earlier papers on adequate sets include the consideration of a more general context, namely countable elementary substructures of $H(\lambda)$ for some $\lambda \geq \omega_2$ of uncountable cofinality, rather than just $H(\omega_2)$. Also we omit the assumption that $2^{\omega_1} = \omega_2$. Since we are not interested in taking initial segments of models as was done in [4], the set Λ which is used to compare models can be taken to be $\omega_2 \cap \mathrm{cof}(\omega_1)$. Some simplifications of the material in [4] follow from these new conventions and from the presence of isomorphisms between models.

Section 2 proves the main result of the paper, that any coherent adequate type forcing preserves CH. Section 3 proves that if CH fails, then any coherent adequate type forcing on $H(\lambda)$, where $\lambda > 2^\omega$ is a cardinal of uncountable cofinality, collapses 2^ω to have size ω_1, preserves $(2^\omega)^+$, and forces CH. Sections 4 and 5 develop the technical machinery for amalgamating conditions over countable elementary substructures; amalgamation is a fundamental tool in side condition forcing for proving cardinal preservation and stronger properties. Sections 6 and 7 give examples of coherent adequate type forcings. In Section 6 we review the poset from [5] for adding a square sequence. In Section 7 we define a coherent adequate type forcing poset which adds a club to a fat stationary subset of ω_2.

Section 8 presents general results for amalgamating coherent adequate sets over elementary substructures of size ω_1. These results are not needed for the present paper, but could be useful for future applications. We show that the forcing poset consisting of finite coherent adequate subsets of $H(\lambda)$ ordered by end-extension is ω_2-c.c.

The general development of coherent adequate sets presented in Sections 1, 4, 5, and 8 is due to Krueger.

Asperó and Mota ([1]) proved recently that for any cardinal $\lambda \geq \omega_2$ of uncountable cofinality, the forcing poset consisting of finite symmetric systems of countable elementary substructures of $H(\lambda)$ ordered by inclusion preserves CH. A symmetric system is similar to a coherent adequate set, except that it does not have the adequate structure. Also, after this work was done, Todorčević pointed out to the authors that in an unpublished result from the 1980s he proved using a different argument that forcing with finite sets of countable elementary substructures of $H(\omega_2)$ with isomorphisms, of the type described at the end of [10], preserves CH and adds an ω_1-Kurepa tree.

Neither of these results show how to force with side conditions together with another finite set of objects to preserve CH, nor do they imply anything regarding

adequate set forcing. By arguments of Miyamoto [8], any coherent adequate type forcing on $H(\lambda)$ adds an ω_1-tree with λ many cofinal branches, for any regular cardinal $\lambda \geq \omega_2$. Thus ω_1-Kurepa trees exist in coherent adequate type forcing extensions.

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1. COHERENT ADEQUATE SETS

In this section we present the basic framework of coherent adequate sets. We will assume throughout the paper that $\lambda \geq \omega_2$ is a fixed cardinal of uncountable cofinality. This implies that any countable subset of $H(\lambda)$ is a member of $H(\lambda)$. We also fix a predicate $Y \subseteq H(\lambda)$, which we assume codes a well-ordering of $H(\lambda)$ among other things.

Let $\mathcal{X}$ denote the set of $N \subseteq H(\lambda)$ such that N is countable and $N \prec (H(\lambda), \in, Y)$. Given a set N in $\mathcal{X}$, let N' denote the set $(N \cap \omega_2) \cup (\lim(N \cap \omega_2))$.

We introduce a way to compare members of $\mathcal{X}$. Let $\Lambda := \omega_2 \cap \operatorname{cof}(\omega_1)$. Note that Λ is a definable subset of $H(\lambda)$.

Definition 1.1. *For $M \in \mathcal{X}$, let Λ_M denote the set of $\beta \in \Lambda$ of the form $\alpha + \omega_1$, for some $\alpha \in M'$.*

Since M is countable, it has countably many limit points. It follows that Λ_M is countable. It is easy to see that $\beta \in \Lambda_M$ iff $\beta = \min(\Lambda \setminus (\sup(M \cap \beta)))$.

Lemma 1.2. *Let $M \in \mathcal{X}$ and $\beta \in \Lambda_M$. If $\beta_0 \in \Lambda \cap \beta$ then $M \cap [\beta_0, \beta) \neq \emptyset$.*

Proof. Let $\beta = \alpha + \omega_1$ for some $\alpha \in M'$. Since there are no ordinals of cofinality ω_1 in the interval (α, β), it is not the case that $\alpha < \beta_0 < \beta$. Hence $\beta_0 \leq \alpha$. If $\alpha \in M$ then we are done. Otherwise α is a limit point of M, and hence has countable cofinality. So $\beta_0 < \alpha$. Since α is a limit point of M, $M \cap (\beta_0, \alpha) \neq \emptyset$. $\qquad\square$

Lemma 1.3. *For M and N in $\mathcal{X}$, $\Lambda_M \cap \Lambda_N$ has a largest element, namely $\sup(M' \cap N') + \omega_1$.*

Proof. Easily $\sup(M' \cap N')$ is in $M' \cap N'$, hence $\sup(M' \cap N') + \omega_1$ is in $\Lambda_M \cap \Lambda_N$. Suppose for a contradiction that there is $\alpha_M \in M'$ and $\alpha_N \in N'$ such that

$$\sup(M' \cap N') + \omega_1 < \alpha_M + \omega_1 = \alpha_N + \omega_1.$$

Let γ_M be the largest ordinal of cofinality ω_1 less than or equal to α_M, and similarly with γ_N. Then easily $\gamma_M = \gamma_N$, and a straightforward argument using elementarity shows that this ordinal is in $M \cap N$. Therefore $\gamma_M < \sup(M' \cap N')$, and hence $\gamma_M + \omega_1 = \alpha_M + \omega_1 \leq \sup(M' \cap N') + \omega_1$, which is a contradiction. $\qquad\square$

Definition 1.4. *For M and N in $\mathcal{X}$, let $\beta_{M,N}$ denote the largest element of $\Lambda_M \cap \Lambda_N$. The ordinal $\beta_{M,N}$ is called the* comparison point *of M and N.*

Lemma 1.5. *Let M and N be in $\mathcal{X}$. Then $M' \cap N' \subseteq \beta_{M,N}$.*

Proof. This follows immediately from Lemma 1.3. $\qquad\square$

If K and M are in $\mathcal{X}$ and $K \subseteq M$, then clearly $\Lambda_K \subseteq \Lambda_M$. It follows that for all N in $\mathcal{X}$, $\max(\Lambda_K \cap \Lambda_N) \leq \max(\Lambda_M \cap \Lambda_N)$. So $\beta_{K,N} \leq \beta_{M,N}$.

We define relations $<$, $\leq$, and $\sim$ on $\mathcal{X}$. Let $M < N$ if $M \cap \beta_{M,N} \in N$. Let $M \sim N$ if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Let $M \leq N$ if either $M < N$ or $M \sim N$.

Definition 1.6. *A subset A of $\mathcal{X}$ is said to be* adequate *if for all M and N in A, either $M < N$, $M \sim N$, or $N < M$.*

Note that any subset of an adequate set is adequate. Also if A is finite and adequate, $M \in \mathcal{X}$, and $A \in M$, then $A \cup \{M\}$ is adequate.

Suppose that $M < N$. Then $M \cap \beta_{M,N} \in N$. But $\beta_{M,N} \in \Lambda_M$ implies that $\beta_{M,N} = \min(\Lambda \setminus (\sup(M \cap \beta_{M,N})))$. Hence $\beta_{M,N}$ is definable in $H(\lambda)$ from $M \cap \beta_{M,N}$. Since N is elementary in $H(\lambda)$, $\beta_{M,N}$ is in N.

Lemma 1.7. *Let M and N be in $\mathcal{X}$ and $\beta \in \Lambda$.*

    (1) *If $M \cap \omega_2 \subseteq \beta$, then $\beta_{M,N} \leq \beta$.*
    (2) *If $\beta < \beta_{M,N}$ and $\{M, N\}$ is adequate, then $M \cap N \cap [\beta, \beta_{M,N}) \neq \emptyset$.*

Proof. (1) Suppose for a contradiction that $\beta < \beta_{M,N}$. Then since $\beta \in \Lambda$ and $\beta_{M,N} \in \Lambda_M$, $M \cap [\beta, \beta_{M,N}) \neq \emptyset$ by Lemma 1.2. This contradicts that $M \cap \omega_2 \subseteq \beta$.

(2) Without loss of generality assume that $M \leq N$. Then $M \cap \beta_{M,N} \subseteq N$. By Lemma 1.2, fix ξ in $M \cap [\beta, \beta_{M,N})$. Then ξ is in $M \cap \beta_{M,N}$ and hence in N. So ξ is in $M \cap N \cap [\beta, \beta_{M,N})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next we define remainder points, which describe the overlap of models past their comparison point.

Definition 1.8. *Let M and N be in $\mathcal{X}$ and assume that $\{M, N\}$ is adequate. Define $R_M(N)$ as the set of β satisfying either:*

    (1) *$M \sim N$ and $\beta = \min(N \setminus \beta_{M,N})$, or*
    (2) *there is $\gamma \in M \setminus \beta_{M,N}$ such that $\beta = \min(N \setminus \gamma)$.*

The set $R_M(N)$ is called the set of *remainder points of N over M*. This set is always finite, since otherwise there would be a common limit point of M and N greater than $\beta_{M,N}$, contradicting Lemma 1.5. For a more detailed proof, see Proposition 2.9 of [4].

Given an adequate set A, define R_A by letting

$$R_A = \bigcup \{R_N(M) : M, N \in A\}.$$

Definition 1.9. *For a given set $S \subseteq \omega_2$, a set $A \subseteq \mathcal{X}$ is (S) adequate if it is adequate and $R_A \subseteq S$.*

Now we move on to coherent adequate sets. We will consider isomorphisms between models in $\mathcal{X}$.

Let M and N be in $\mathcal{X}$ and let $\sigma : M \to N$. We say that σ is an *isomorphism* if σ is a bijection and for all a and b in M,

    • $a \in b$ iff $\sigma(a) \in \sigma(b)$;
    • $a \in Y$ iff $\sigma(a) \in Y$.

In other words, σ is an isomorphism if it is an isomorphism in the usual model theoretic sense between the structures $(M, \in, Y \cap M)$ and $(N, \in, Y \cap N)$. We say that M and N are *isomorphic* if there exists an isomorphism from M to N. Note that the isomorphism relation is an equivalence relation.

For a model M in $\mathcal{X}$, the elementarity of M in $H(\lambda)$ implies that M satisfies the axiom of extensionality. It follows that $(M, \in, Y \cap M)$ is isomorphic to a unique transitive structure $(\overline{M}, \in, \overline{Y})$ by a unique isomorphism $\sigma_M : M \to \overline{M}$ given by

the recursive equation $\sigma_M(a) = \sigma_M[a \cap M]$. Note that if $a \in M$ is countable, then $a \subseteq M$, so $\sigma_M(a) = \sigma_M[a]$.

By the uniqueness of the transitive collapse, a standard argument shows that M and N in $\mathcal{X}$ are isomorphic iff the structures $(M, \in, Y \cap M)$ and $(N, \in, Y \cap N)$ have the same transitive collapse. In that case, $\sigma_N^{-1} \circ \sigma_M$ is an isomorphism from M to N, which we denote by $\sigma_{M,N}$. Also the uniqueness of the transitive collapsing map easily implies that $\sigma_{M,N}$ is the unique isomorphism from M to N. Note that this map satisfies that if $a \in M$ is countable, then $\sigma_{M,N}(a) = \sigma_{M,N}[a]$. Also by the uniqueness of isomorphisms, if M, N, and P are isomorphic, then $\sigma_{M,P} = \sigma_{N,P} \circ \sigma_{M,N}$.

Lemma 1.10. *Let M, N and K be in $\mathcal{X}$ such that M and N are isomorphic and $K \in M$. Let $L := \sigma_{M,N}(K)$. Then:*

    (1) *$L \in \mathcal{X}$;*
    (2) *K and L are isomorphic and $\sigma_{K,L} = \sigma_{M,N} \restriction K$.*

Proof. (1) Since K is countable, $L = \sigma_{M,N}(K) = \sigma_{M,N}[K]$. Hence $L \subseteq N$. As $(N, \in, N \cap Y) \prec (H(\lambda), \in, Y)$, it suffices to show that $\mathfrak{L} := (L, \in, L \cap Y) \prec (N, \in, N \cap Y)$. Let $\mathfrak{K} := (K, \in, K \cap Y)$. By the elementarity of M and N, the predicates $K \cap Y$ are $L \cap Y$ are in M and N respectively. Let $b_1, \ldots, b_k$ be in L and let $\varphi(x_1, \ldots, x_k)$ be a formula in the language of the structure $(H(\lambda), \in, Y)$. Let $a_i := \sigma_{N,M}(b_i)$ for $i = 1, \ldots, k$. Then $(L, \in, L \cap Y) \models \varphi[b_1, \ldots, b_k]$ iff $N \models \ulcorner\varphi\urcorner^{\mathfrak{L}}[b_1, \ldots, b_k]$ iff $M \models \ulcorner\varphi\urcorner^{\mathfrak{K}}[a_1, \ldots, a_k]$ iff $M \models \varphi[a_1, \ldots, a_k]$ iff $N \models \varphi[b_1, \ldots, b_k]$, where the third equivalence follows from the fact that $\mathfrak{K}$ is an elementary substructure of $(M, \in, M \cap Y)$.

(2) Since $L = \sigma_{M,N}(K) = \sigma_{M,N}[K]$, $\sigma_{M,N} \restriction K$ is a bijection from K to L. It is obvious that $\sigma_{M,N} \restriction K$ preserves the predicates $\in$ and Y since $\sigma_{M,N}$ does. So $\sigma_{M,N} \restriction K$ is an isomorphism of K to L. By the uniqueness of isomorphisms, $\sigma_{M,N} \restriction K = \sigma_{K,L}$. $\qquad\square$

Lemma 1.11. *Suppose that M and N are in $\mathcal{X}$ and are isomorphic. Let $\sigma := \sigma_{M,N}$. Assume that K and L are in $M \cap \mathcal{X}$. Then:*

    (1) *$\sigma(\beta_{K,L}) = \beta_{\sigma(K),\sigma(L)}$;*
    (2) *$K < L$ iff $\sigma(K) < \sigma(L)$;*
    (3) *$K \sim L$ iff $\sigma(K) \sim \sigma(L)$;*
    (4) *$K \leq L$ iff $\sigma(K) \leq \sigma(L)$;*
    (5) *$\sigma(R_K(L)) = R_{\sigma(K)}(\sigma(L))$;*
    (6) *if $\{K, L\}$ is adequate then $\{\sigma(K), \sigma(L)\}$ is adequate.*

Proof. The lemma follows from the fact for any P and Q in $\mathcal{X}$, the objects and relations $\beta_{P,Q}$, $P < Q$, $P \leq Q$, and $R_P(Q)$ are definable in $H(\lambda)$ from P and Q. $\quad\square$

We now introduce an additional requirement on isomorphisms. Let us say that M and N in $\mathcal{X}$ are *strongly isomorphic* if they are isomorphic and for all $a \in M \cap N$, $\sigma_{M,N}(a) = a$. We write $M \cong N$ to indicate that M and N are strongly isomorphic.

Lemma 1.12. *Let M, N and K be in $\mathcal{X}$ such that M and N are strongly isomorphic and $K \in M$. Let $L := \sigma_{M,N}(K)$. Then K and L are strongly isomorphic.*

Proof. By Lemma 1.10, K and L are isomorphic and $\sigma_{K,L} = \sigma_{M,N} \restriction K$. Let $a \in K \cap L$ be given. Then $a \in M \cap N$. So $\sigma_{K,L}(a) = \sigma_{M,N}(a) = a$. $\qquad\square$

Lemma 1.13. *Suppose that N, N', and N^* are in $\mathcal{X}$ and are strongly isomorphic. Let $a \in N' \cap N^*$. Then $\sigma_{N',N}(a) = \sigma_{N^*,N}(a)$.*

Proof. Since $a \in N' \cap N^*$, $\sigma_{N',N^*}(a) = a$. So $\sigma_{N',N}(a) = \sigma_{N^*,N}(\sigma_{N',N^*}(a)) = \sigma_{N^*,N}(a)$. $\qquad\square$

Lemma 1.14. *Let N, N', N^*, K, and L be in $\mathcal{X}$ such that N, N', and N^* are strongly isomorphic, K and L are strongly isomorphic, and K and L are in N. Let $M := \sigma_{N,N'}(K)$ and $P := \sigma_{N,N^*}(L)$. Then M and P are strongly isomorphic.*

Proof. By Lemma 1.10, K and M are isomorphic and L and P are isomorphic. Since K and L are isomorphic and being isomorphic is an equivalence relation, M and P are isomorphic. To show that M and P are strongly isomorphic, let $a \in M \cap P$ and we will show that $\sigma_{M,P}(a) = a$. Let $b := \sigma_{N',N}(a)$. Since $a \in M \cap P$, $a \in N' \cap N^*$. By Lemma 1.13, $\sigma_{N^*,N}(a) = \sigma_{N',N}(a) = b$. Now $a \in M$ implies that $b \in K$, and $a \in P$ implies that $b \in L$. So $b \in K \cap L$. Since K and L are strongly isomorphic, $\sigma_{K,L}(b) = b$. So $\sigma_{M,P}(a) = \sigma_{L,P}(\sigma_{K,L}(\sigma_{M,K}(a))) = \sigma_{N,N^*}(\sigma_{K,L}(\sigma_{N',N}(a))) = \sigma_{N,N^*}(\sigma_{K,L}(b)) = \sigma_{N,N^*}(b) = a$. $\qquad\square$

Definition 1.15. *Let A be a subset of $\mathcal{X}$. We say that A is* coherent adequate *if A is adequate, and for all M and N in A:*

(1) *if $M \sim N$, then M and N are strongly isomorphic;*
(2) *if $M < N$, then there exists $N' \in A$ isomorphic to N such that $M \in N'$;*
(3) *if M and N are isomorphic and $K \in M \cap A$, then $\sigma_{M,N}(K) \in A$.*

Let M and N be in $\mathcal{X}$. Since $\omega_1 \leq \beta_{M,N}$, $M \sim N$ implies that $M \cap \omega_1 = N \cap \omega_1$. Also if $M < N$, then $M \cap \omega_1$ is an initial segment of $M \cap \beta_{M,N}$ and hence is in N. It follows that if $\{M,N\}$ is adequate, then $M \sim N$ iff $M \cap \omega_1 = N \cap \omega_1$. Also if M and N are isomorphic, then since ω_1 is definable in $H(\lambda)$, $\sigma_{M,N}(\omega_1) = \omega_1$. It easily follows that $M \cap \omega_1 = N \cap \omega_1$.

As a consequence of these observations, if A is coherent adequate and M and N are in A, then the following are equivalent:

- $M \sim N$;
- $M \cap \omega_1 = N \cap \omega_1$;
- M and N are isomorphic;
- M and N are strongly isomorphic.

Moreover, $M < N$ iff $M \cap \omega_1 < N \cap \omega_1$.

Given a set $S \subseteq \omega_2$, a set $A \subseteq \mathcal{X}$ is (S) *coherent adequate* if A is coherent adequate and $R_A \subseteq S$.

Lemma 1.16. *Let A be a finite coherent adequate set.*

(1) *If $M \in \mathcal{X}$ and $A \in M$, then $A \cup \{M\}$ is coherent adequate.*
(2) *If $M \in A$, then $M \cap A$ is coherent adequate.*

Proof. (1) We already observed that $A \cup \{M\}$ is adequate, and the coherent properties are immediate. (2) Requirements (1) and (3) in the definition of coherent adequate are obvious. For requirement (2), suppose that K and N are in $A \cap M$ and $K < N$. Since A is coherent, there is $N' \in A$ isomorphic to N such that $K \in N'$. If $N' \in M$ then we are done, so assume not. As $N' \cap \omega_1 = N \cap \omega_1 < M \cap \omega_1$, $N' < M$. So there is $M' \in A$ isomorphic to M with $N' \in M'$. Let $N^* = \sigma_{M',M}(N')$, which is in $M \cap A$. Since $K \in N'$ and $N' \in M'$, $K \in M'$. But also $K \in M$, so

$\sigma_{M',M}(K) = K$. Since $K \in N'$, $K = \sigma_{M',M}(K) \in \sigma_{M,M'}(N') = N^*$. So $K \in N^*$, N^* is isomorphic to N, and $N^* \in M \cap A$. $\qquad\square$

Lemma 1.17. *Let A be a finite coherent adequate set. Suppose that $\{M_0, \ldots, M_k\}$ is adequate and consists of strongly isomorphic sets. Assume that $A \in M_0 \cap \cdots \cap M_k$. Then $A \cup \{M_0, \ldots, M_k\}$ is coherent adequate.*

Proof. It is obvious that $A \cup \{M_0, \ldots, M_k\}$ is adequate and satisfies requirements (1) and (2) in the definition of coherent. Requirement (3) follows from the fact that for any $i, j \leq k$, if $K \in M_i \cap A$, then $K \in M_j$, and therefore $\sigma_{M_i,M_j}(K) = K \in A$. $\qquad\square$

Lemma 1.18. *Let M and N be in $\mathcal{X}$ and assume that M and N are strongly isomorphic. Then $M \cap P_{\omega_1}(M \cap N) \subseteq N$. In particular, if $\{M, N\}$ is adequate and M and N are strongly isomorphic, then $M \cap P_{\omega_1}(\beta_{M,N}) \subseteq N$.*

Proof. Let $a \in M \cap P_{\omega_1}(M \cap N)$ be given. Since a is countable and $\sigma_{M,N}$ is the identity on $M \cap N$, $\sigma_{M,N}(a) = \sigma_{M,N}[a] = a$. So $a \in N$. If in addition we know that $\{M, N\}$ is adequate, then $M \sim N$. So if $a \in M \cap P_{\omega_1}(\beta_{M,N})$, then a is a countable subset of $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Hence $a \in M \cap P_{\omega_1}(M \cap N)$ and therefore $a \in N$. $\qquad\square$

Lemma 1.19. *Let A be a coherent adequate set. Suppose that M and N are in A and $M \leq N$. Then $M \cap P_{\omega_1}(\beta_{M,N}) \subseteq N$.*

Proof. If M and N are strongly isomorphic, then we are done by the preceding lemma. Assume that $M < N$. Since A is coherent, fix $N' \in A$ isomorphic to N such that $M \in N'$. Then N and N' are strongly isomorphic. Consider $a \in M \cap P_{\omega_1}(\beta_{M,N})$. Then $a \in N' \cap P_{\omega_1}(\beta_{M,N})$. Since $M \in N'$, $\beta_{M,N} \leq \beta_{N,N'}$. So $a \in N' \cap P_{\omega_1}(\beta_{N',N})$. As N and N' are strongly isomorphic, $a \in N$ by the preceding lemma. $\qquad\square$

2. Preserving CH

Fix for the remainder of this section a set $S \subseteq \omega_2$ such that $S \cap \text{cof}(\omega_1)$ is stationary and a set $\mathcal{Y} \subseteq \mathcal{X}$ which is stationary in $P_{\omega_1}(H(\lambda))$. Also assume that $\mathcal{Y}$ is closed under isomorphisms, which means that whenever M and N are in $\mathcal{Y}$ and are isomorphic, and $K \in M \cap \mathcal{Y}$, then $\sigma_{M,N}(K) \in \mathcal{Y}$. Note that by Lemma 1.10, the set $\mathcal{X}$ itself is closed under isomorphisms.

A forcing poset $\mathbb{P}$ is said to be an $(S, \mathcal{Y})$ *coherent adequate type forcing* if there exists a natural number m such that $\mathbb{P}$ consists of conditions of the form $(x_0, \ldots, x_m, A)$ satisfying:

(I) $x_0, \ldots, x_m$ are finite subsets of $H(\lambda)$;
(II) A is a finite (S) coherent adequate subset of $\mathcal{Y}$;
(III) if $(y_0, \ldots, y_m, B) \leq (x_0, \ldots, x_m, A)$, N and N' are isomorphic sets in B, and $(x_0, \ldots, x_m, A) \in N$, then $\sigma_{N,N'}((x_0, \ldots, x_m, A))$ is in $\mathbb{P}$, and $(y_0, \ldots, y_m, B) \leq \sigma_{N,N'}((x_0, \ldots, x_m, A))$;
(IV) if $M_0, \ldots, M_n$ are isomorphic sets in $\mathcal{Y}$ such that $\{M_0, \ldots, M_n\}$ is (S) coherent adequate and $(x_0, \ldots, x_m, A) \in M_0 \cap \cdots \cap M_n$, then there is a condition $(y_0, \ldots, y_m, B) \leq (x_0, \ldots, x_m, A)$ such that $M_0, \ldots, M_n \in B$;
(V) for all $M \in A$, $(x_0, \ldots, x_m, A)$ is strongly M-generic.

Regarding (V), recall that a condition q is strongly M-generic if for any set D which is dense in the forcing poset $M \cap \mathbb{P}$, D is predense in $\mathbb{P}$ below q.

We say that $\mathbb{P}$ is a *coherent adequate type forcing* if $\mathbb{P}$ is an $(\omega_2, \mathcal{X})$ coherent adequate type forcing. Define (S) coherent adequate and $(\mathcal{Y})$ coherent adequate type forcings to mean $(S, \mathcal{X})$ and $(\omega_2, \mathcal{Y})$ coherent adequate type forcings respectively. We interpret the above definition to include the possibility that the sequence $x_0, \ldots, x_m$ in a condition has length 0, in which case conditions are just $(S, \mathcal{Y})$ coherent adequate sets (although the ordering need not be just reverse inclusion).

Lemma 2.1. *Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate forcing. Then $\mathbb{P}$ preserves ω_1.*

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name for a function from ω to ω_1 and let p be a condition. We will find $q \le p$ which forces that the range of $\dot{f}$ is bounded. Fix $\chi \ge \lambda$ regular such that $\dot{f} \in H(\chi)$. Since $\mathcal{Y}$ is stationary, we can find N^* a countable elementary substructure of $H(\chi)$ with $\mathbb{P}$, $\dot{f}$, and p in N^* and $N := N^* \cap H(\lambda)$ in $\mathcal{Y}$.

By property (IV), there is $q = (y_0, \ldots, y_m, B) \le p$ such that $N \in B$. By property (V), q is strongly N-generic. It is easy to see that for all $n < \omega$, the set of $s \in N \cap \mathbb{P}$ which decide the value of $\dot{f}(n)$ is dense in $N \cap \mathbb{P}$. It follows that for all $n < \omega$, if $r \le q$ decides the value of $\dot{f}(n)$, then that value is a member of N. So q forces that $\dot{f}$ is bounded. $\qquad\square$

We will prove that assuming CH, any $(S, \mathcal{Y})$ coherent adequate type forcing preserves ω_2 and CH. The proof will use the following technical result.

Lemma 2.2. *(CH) Suppose that $R_0, \ldots, R_k$ are subsets of $H(\lambda)$. Then for any set $z \in H(\lambda)$, there are M and N in $\mathcal{Y}$ satisfying the following:*
  (1) *$z \in M \cap N$;*
  (2) *$\{M, N\}$ is (S) coherent adequate;*
  (3) *the structures $(M, \in, Y \cap M, R_0 \cap M, \ldots, R_k \cap M)$ and $(N, \in, Y \cap N, R_0 \cap N, \ldots, R_k \cap N)$ are elementary in $(H(\lambda), \in, Y, R_0, \ldots, R_k)$ and are isomorphic;*
  (4) *$M \cap \omega_2 \subseteq \min(N \setminus \beta_{M,N})$ and $\sigma_{M,N}(\min(M \setminus \beta_{M,N})) = \min(N \setminus \beta_{M,N})$.*

Proof. Since $S \cap \operatorname{cof}(\omega_1)$ is stationary and CH holds, we can fix N^* satisfying:
  (1) $N^* \prec H(2^\lambda)$;
  (2) $|N^*| = \omega_1$;
  (3) $Y, z, S, \mathcal{Y}, R_0, \ldots, R_k$ are in N^*;
  (4) $\beta^* := N^* \cap \omega_2 \in S$;
  (5) $(N^*)^\omega \subseteq N^*$;

As $\mathcal{Y}$ is stationary, we can fix N in $\mathcal{Y}$ such that $z, \beta^* \in N$ and

$$(N, \in, Y \cap N, R_0 \cap N, \ldots, R_k \cap N) \prec (H(\lambda), \in, Y, R_0, \ldots, R_k).$$

Let $\mathfrak{N}$ denote the structure $(N, \in, Y \cap N, R_0 \cap N, \ldots, R_k \cap N)$, let $\overline{\mathfrak{N}}$ denote its transitive collapse, and as usual let σ_N denote the transitive collapsing map. Let T be the relation defined by letting $T(a, b)$ hold if $a \in N \cap N^*$ and $\sigma_N(a) = b$. By CH, $H(\omega_1)$ has size ω_1. Since $(N^*)^\omega \subseteq N^*$, $H(\omega_1)$ is a subset of N^*. It follows that $\overline{\mathfrak{N}}$ is a member and a subset of N^*. Also N^* contains the sets $N \cap N^*$ and $N \cap \beta^*$ as members. It follows that the relation T is a subset of N^*, and since it is countable it is a member of N^*.

The objects N and β^* witness the statement, satisfied by the structure $H(2^\lambda)$, that there exist M and β satisfying:

   (a) M is in $\mathcal{Y}$;
   (b) $\beta \in S \cap \mathrm{cof}(\omega_1)$;
   (c) z and β are in M;
   (d) $N \cap \beta^* = M \cap \beta$;
   (e) $N \cap N^* \subseteq M$;
   (f) $(M, \in, Y, R_0 \cap M, \ldots, R_k \cap M) \prec (H(\lambda), \in, Y, R_0, \ldots, R_k)$;
   (g) the transitive collapse of the structure $(M, \in, Y \cap M, R_0 \cap M, \ldots, R_k \cap M)$ is equal to $\overline{\mathfrak{N}}$;
   (h) $T(a, b)$ implies that $\sigma_M(a) = b$.

Since the parameters which appear in the above statement are all members of N^*, by elementarity we can fix M and β in N^* satisfying the same statement.

Let us prove that M and N are as desired. We know that M and N are in $\mathcal{Y}$ and $z \in M \cap N$. Also the structures $(M, \in, Y \cap M, R_0 \cap M, \ldots, R_k \cap M)$ and $(N, \in, Y \cap N, R_0 \cap N, \ldots, R_k \cap N)$ are elementary in $(H(\lambda), \in, Y, R_0, \ldots, R_k)$. And since they have the same transitive collapse, they are isomorphic.

We claim that M and N are strongly isomorphic. So let $a \in M \cap N$ be given, and we show that $\sigma_{N,M}(a) = a$. Since $M \in N^*$, it follows that $a \in N \cap N^*$. Let $b := \sigma_N(a)$. Then $T(a, b)$ holds. By (h), $\sigma_M(a) = b$. It follows that $\sigma_{N,M}(a) = \sigma_M^{-1}(\sigma_N(a)) = \sigma_M^{-1}(b) = a$.

We claim that $\beta_{M,N} \leq \beta$. Since $M \in N^*$, $\Lambda_M \subseteq N^*$. But $\beta_{M,N} \in \Lambda_M$, so $\beta_{M,N} \in N^* \cap \omega_2 = \beta^*$. This shows that $\beta_{M,N} \leq \beta^*$. Let $\xi := \sup(N \cap \beta_{M,N})$. As $\beta_{M,N} \leq \beta^*$, ξ is a limit point of $N \cap \beta^*$. Since $\beta_{M,N} \in \Lambda_N$, $\beta_{M,N} = \min(\Lambda \setminus \xi)$. Now $N \cap \beta^* = M \cap \beta$ implies that $N \cap \beta^* \subseteq \beta$. Since ξ is a limit point of $N \cap \beta^*$, $\xi < \beta$. As $\beta_{M,N} = \min(\Lambda \setminus \xi)$ and $\beta \in \Lambda$, $\beta_{M,N} \leq \beta$.

Now $M \cap \beta = N \cap \beta^*$ and $\beta_{M,N} \leq \beta$ imply that

   (i) $M \sim N$;
   (ii) $\min(M \setminus \beta_{M,N}) \geq \beta$;
   (iii) $\min(N \setminus \beta_{M,N}) \geq \beta^*$.

Statement (i) is immediate. For (ii), if $\tau \in M \setminus \beta_{M,N}$, then $\tau \in M \setminus N$. As $M \cap \beta \subseteq N$, $\tau \geq \beta$. The proof of (iii) is similar.

By (i), $\{M, N\}$ is adequate. And since M and N are strongly isomorphic, $\{M, N\}$ is coherent adequate. As $\beta \in M$ and $\beta^* \in N$, (ii) and (iii) imply that $\min(M \setminus \beta_{M,N}) = \beta$ and $\min(N \setminus \beta_{M,N}) = \beta^*$. Also $M \in N^*$ implies that $M \cap \omega_2 \subseteq \beta^*$. This easily implies that $R_M(N) = \{\beta^*\}$ and $R_N(M) = \{\beta\}$. As β and β^* are in S, $\{M, N\}$ is (S) coherent adequate.

Since β and β^* are the first elements of M and N above their common intersection, clearly $\sigma_{M,N}(\beta) = \beta^*$. Hence $\sigma_{M,N}(\min(M \setminus \beta_{M,N})) = \min(N \setminus \beta_{M,N})$. Finally, $M \cap \omega_2 \subseteq \beta^* = \min(N \setminus \beta_{M,N})$. $\qquad\square$

Proposition 2.3. *(CH) Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate type forcing. If p is a condition which forces that $\langle \dot{f}_i : i < \omega_2^V \rangle$ is a sequence of functions from ω to ω, then there is $q \leq p$ and $i < j$ such that q forces that $\dot{f}_i = \dot{f}_j$.*

Proof. Define a relation R on $H(\lambda)$ by letting $R(z, i, n, m)$ if $z \in \mathbb{P}$ and $z \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$. By Lemma 2.2, we can fix M and N in $\mathcal{Y}$ satisfying:

   (1) $p \in M \cap N$;
   (2) $\{M, N\}$ is (S) coherent adequate;
   (3) the structures $(M, \in, Y \cap M, R \cap M)$ and $(N, \in, Y \cap N, R \cap N)$ are elementary in $(H(\lambda), \in, Y, R)$ and are isomorphic;

(4) $M \cap \omega_2 \subseteq \min(N \setminus \beta_{M,N})$ and $\sigma_{M,N}(\min(M \setminus \beta_{M,N})) = \min(N \setminus \beta_{M,N})$.

Let $\sigma := \sigma_{M,N}$. By the uniqueness of isomorphisms, σ is an isomorphism between the structures described in (3) above. Let $i := \min(M \setminus \beta_{M,N})$ and $j := \min(N \setminus \beta_{M,N})$. Then $i < j$ and $\sigma(i) = j$.

We claim that for all $z \in M \cap \mathbb{P}$ and integers n and m, if $z \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$ then $\sigma(z) \Vdash_{\mathbb{P}} \dot{f}_j(n) = m$. For assume that $z \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$. Then $R(z, i, n, m)$ holds. Since z, i, n, and m are in M, $\sigma(i) = j$, and σ is an isomorphism, $R(\sigma(z), j, n, m)$ holds. By definition of R, this means that $\sigma(z) \Vdash_{\mathbb{P}} \dot{f}_j(n) = m$.

Since M and N are isomorphic sets in $\mathcal{Y}$, $\{M, N\}$ is (S) coherent adequate, and $p \in M \cap N$, by property (IV) in the definition of an $(S, \mathcal{Y})$ coherent adequate type forcing, there is a condition $q = (y, B)$ below p such that $M, N \in B$. By property (V), q is strongly M-generic.

We claim that q forces that $\dot{f}_i = \dot{f}_j$, which completes the proof. So assume for a contradiction that there exists $r \leq q$ and $n < \omega$ such that $r \Vdash_{\mathbb{P}} \dot{f}_i(n) \neq \dot{f}_j(n)$.

Let D be the set of conditions w in $\mathbb{P} \cap M$ such that for some m, $w \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$. We claim that D is dense in $\mathbb{P} \cap M$. So let $v \in \mathbb{P} \cap M$ be given. Clearly there exists w and m such that $w \leq v$ and $w \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$, and hence $R(w, i, n, m)$ holds. Since v, i, and n are in M and M is elementary in $(H(\lambda), \in, Y, R)$, there is $w \leq v$ in $\mathbb{P} \cap M$ and m such that $R(w, i, n, m)$ holds. Then w is an extension of v in D.

As q is strongly M-generic, D is predense below q. In particular, there is $w \in D$ such that w is compatible with r. Fix $s \leq w, r$. So $s \leq w$, M and N are isomorphic models in A_s, and $w \in M$. By Property (III) in the definition of an $(S, \mathcal{Y})$ coherent adequate type forcing, $s \leq \sigma(w)$. Since $w \Vdash_{\mathbb{P}} \dot{f}_i(n) = m$, $\sigma(w) \Vdash_{\mathbb{P}} \dot{f}_j(n) = m$ by the claim above. As s extends both w and $\sigma(w)$, s forces that $\dot{f}_i(n) = \dot{f}_j(n)$. But this contradicts that $s \leq r$, since r forces that $\dot{f}_i(n) \neq \dot{f}_j(n)$. $\square$

Corollary 2.4. *(CH) Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate type forcing. Then $\mathbb{P}$ preserves ω_2 and CH.*

Proof. The statement that $\mathbb{P}$ preserves CH is immediate from the proposition. Suppose for a contradiction that $\mathbb{P}$ does not preserve ω_2, and let p be a condition which forces that $|\omega_2^V| = \omega_1$. Then we can find a sequence of names which p forces is an enumeration of ω_1 many distinct functions from ω to ω in order type ω_2^V, contradicting the proposition. $\square$

3. Collapsing the Continuum

In this section we analyze what happens when we force with a coherent adequate type forcing over a model in which CH fails. We will prove that in this context, the cardinal $(2^\omega)^V$ will be collapsed to have size ω_1, its successor in V will become ω_2, and CH will hold in the extension .

Let λ be any cardinal with uncountable cofinality such that $2^\omega < \lambda$. Let $\langle r_i : i < 2^\omega \rangle$ be an enumeration of the power set of ω such that $r_i \neq r_j$ for all $i < j < 2^\omega$. Moreover, assume that $\langle r_i : i < 2^\omega \rangle$ is the first such enumeration according to the well-ordering of $H(\lambda)$ coded by the predicate Y. It follows that Y codes the relation Z, where $Z(i, n)$ holds if $i < 2^\omega$ and $n \in r_i$.

Lemma 3.1. *Suppose that M and N are in $\mathcal{X}$ and are isomorphic. Then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Therefore $M \cap 2^\omega = N \cap 2^\omega$.*

Proof. Let $\alpha \in M \cap 2^\omega$ be given. Since $r_i \neq r_j$ for all $i < j < 2^\omega$, it suffices to show that $r_\alpha = r_{\sigma_{M,N}(\alpha)}$. So let $n < \omega$ be given. Then $n \in r_\alpha$ iff $M \models Z(\alpha, n)$ iff $N \models Z(\sigma_{M,N}(\alpha), n)$ iff $n \in r_{\sigma_{M,N}(\alpha)}$. $\qquad\square$

Theorem 3.2. *Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate type forcing. Let μ be the cardinal 2^ω. Then $\mathbb{P}$ collapses μ to have size ω_1.*

Proof. Suppose that $p = (x_0, \ldots, x_k, A)$ is a condition, M and N are in A, and $M < N$. We claim that $M \cap \mu \in N$. By coherence, fix N' in A such that N and N' are isomorphic and $M \in N'$. Then $M \cap \mu \in N'$. By Lemma 3.1, $\sigma_{N',N}(M \cap \mu) = \sigma_{N',N}[M \cap \mu] = M \cap \mu$. Hence $M \cap \mu$ is in N.

Let G be a generic filter for $\mathbb{P}$. Let $J := \{N : \exists (x_0, \ldots, x_k, A) \in G \ (N \in A)\}$. Then for all M and N in J, if $M \cap \omega_1 = N \cap \omega_1$ then M and N are isomorphic. For in that case the conditions in G witnessing and M and N are in J have a common extension in G, and the set of models in this condition is coherent adequate. By Lemma 3.1, $M \cap \mu = N \cap \mu$. Similarly, if M and N are in J and $M \cap \omega_1 < N \cap \omega_1$, then we can find a condition $(x_0, \ldots, x_k, A)$ in G such that M and N are in A. By the previous paragraph, $M \cap \mu \in N$. In particular, $M \cap \mu \subseteq N \cap \mu$.

It follows that $K := \{N \cap \mu : N \in J\}$ is well ordered by the subset relation in order type ω_1. Using property (IV) in the definition of a coherent adequate type forcing, an easy density argument shows that for any $i < \mu$, there is N in J such that $i \in N \cap \mu$. It follows that $\bigcup K = \mu$. Hence in $V[G]$, μ is the union of ω_1 many countable sets and therefore has size ω_1. $\qquad\square$

Lemma 3.3. *Assume that $\omega_2 \leq 2^\omega < \lambda$. Suppose that $R_0, \ldots, R_k$ are subsets of $H(\lambda)$. Then for any set $z \in H(\lambda)$, there are M and N in $\mathcal{Y}$ satisfying the following:*

(1) *$z \in M \cap N$;*
(2) *$\{M, N\}$ is (S) coherent adequate;*
(3) *the structures $(M, \in, Y \cap M, R_0 \cap M, \ldots, R_k \cap M)$ and $(N, \in, Y \cap N, R_0 \cap N, \ldots, R_k \cap N)$ are elementary in $(H(\lambda), \in, Y, R_0, \ldots, R_k)$ and are isomorphic;*
(4) *there exists $\alpha \in M \cap (2^\omega)^+$ and $\beta \in N \cap (2^\omega)^+$ such that $\alpha \neq \beta$ and $\sigma_{M,N}(\alpha) = \beta$.*

Proof. For each $i < (2^\omega)^+$ fix N_i in $\mathcal{Y}$ such that z and i are in N_i and N_i is an elementary substructure of $(H(\lambda), \in, Y, R_0, \ldots, R_k)$. This is possible since $\mathcal{Y}$ is stationary. Let $\mathfrak{N}_i$ denote the structure $(N_i, \in, Y \cap N_i, R_0 \cap N_i, \ldots, R_k \cap N_i)$ and let $\overline{\mathfrak{N}}_i$ denote its transitive collapse. Since $H(\omega_1)$ has size 2^ω, we can fix a cofinal set $P \subseteq (2^\omega)^+$ such that for all $i < j$ in P, $\overline{\mathfrak{N}}_i = \overline{\mathfrak{N}}_j$. It follows that $\mathfrak{N}_i$ and $\mathfrak{N}_j$ are isomorphic.

By the Δ-system lemma, there is a cofinal set $P' \subseteq P$ and a countable set z such that for all $i < j$ in P', $N_i \cap N_j = z$. As there are 2^ω many possibilities for $\sigma_{N_i} \upharpoonright z$ for $i \in P'$, where σ_{N_i} is the transitive collapsing map of N_i, we can find a cofinal set $P'' \subseteq P'$ such that for all $i < j$ in P'', $\sigma_{N_i} \upharpoonright z = \sigma_{N_j} \upharpoonright z$. It follows that if $a \in N_i \cap N_j = z$, then $\sigma_{N_i,N_j}(a) = \sigma_{N_j}^{-1}(\sigma_{N_i}(a)) = a$. So N_i and N_j are strongly isomorphic.

Let M be equal to N_i for some fixed $i \in P''$. Then $\sup(M \cap (2^\omega)^+) < (2^\omega)^+$ since M is countable. So we can find $i < j$ in P'' such that $\sup(M \cap (2^\omega)^+) < j$. Let $N := N_j$. We will prove that M and N are as desired.

Properties (1) and (3) are immediate. Since $\omega_2 \leq 2^\omega$ and M and N are isomorphic, $M \cap \omega_2 = N \cap \omega_2$ by Lemma 3.1. So trivially $\{M, N\}$ is adequate. Also $R_M(N)$ and $R_N(M)$ are empty, so $\{M, N\}$ is (S) adequate. As M and N are strongly isomorphic, $\{M, N\}$ is (S) coherent adequate. This verifies property (2). For (4), let $\beta := j$. Then $\sup(M \cap (2^\omega)^+) < \beta$. Since $(2^\omega)^+$ is either equal to λ or definable in $H(\lambda)$, $\alpha := \sigma_{N,M}(\beta)$ is in $M \cap (2^\omega)^+$ and hence is below β. So $\alpha \neq \beta$ and $\sigma_{M,N}(\alpha) = \beta$. □

Proposition 3.4. *Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate type forcing. Let $\mu := (2^\omega)^+$. If p is a condition which forces that $\langle \dot{f}_i : i < \mu \rangle$ is a sequence of functions from ω to ω, then there is $q \leq p$ and $i < j$ such that q forces that $\dot{f}_i = \dot{f}_j$.*

The proof is nearly identical to the proof of Proposition 2.3, replacing ω_2^V with μ and replacing references to Lemma 2.2 with references to Lemma 3.3.

Corollary 3.5. *Let $\mathbb{P}$ be an $(S, \mathcal{Y})$ coherent adequate type forcing. Then $\mathbb{P}$ collapses $(2^\omega)^V$ to have size ω_1, forces that the successor of $(2^\omega)^V$ in V is equal to ω_2, and forces CH.*

The proof of the corollary follows from Proposition 3.4 in the same way that Corollary 2.4 follows from Proposition 2.3.

4. Some Lemmas on Closure

The next two sections will develop the technology needed to amalgamate coherent adequate style forcings over countable elementary substructures. Amalgamation is a fundamental tool in side condition forcing for proving preservation of cardinals and stronger properties; it will be used heavily in Section 7. There are several ways in which this development goes beyond the analogous results in [5]. Besides the more general context of $H(\lambda)$, we also give an analysis of the remainder points produced under such amalgamation, and work out general results on closure. The present section handles the topic of closure.[1]

Let A be a subset of $\mathcal{X}$ and let x be a finite subset of $H(\lambda)$. We say that the pair (x, A) is *closed* if whenever N and N' are isomorphic sets in A and $a \in x \cap N$, then $\sigma_{N,N'}(a) \in x$.

Lemma 4.1. *Let A be a finite coherent adequate set and x a finite subset of $H(\lambda)$. Let y be the set*

$$x \cup \{\sigma_{M,M'}(a) : M, M' \in A, \ M \cong M', \ a \in x \cap M\}.$$

Then (y, A) is closed.

Proof. Let N and N' be isomorphic sets in A and $a \in y \cap N$. We will prove that $\sigma_{N,N'}(a) \in y$. If $a \in x$, then $\sigma_{N,N'}(a) \in y$ by the definition of y. Otherwise there are M and M' in A which are isomorphic and b in $x \cap M$ such that $a = \sigma_{M,M'}(b)$. Then a is in $M' \cap N$.

Case 1: $M' < N$. Fix $N^* \in A$ isomorphic to N with $M' \in N^*$. Then $a \in N^* \cap N$, so $\sigma_{N^*,N'}(a) = \sigma_{N,N'}(a)$. Let $M_1 = \sigma_{N^*,N'}(M')$. Then $M_1 \in A$ and $\sigma_{M',M_1} = \sigma_{N^*,N'} \restriction M'$ by Lemma 1.10. Now $\sigma_{M,M_1}(b)$ is in y by the definition of y. But $\sigma_{M,M_1}(b) = \sigma_{M',M_1}(\sigma_{M,M'}(b)) = \sigma_{N^*,N'}(a) = \sigma_{N,N'}(a)$. Hence $\sigma_{N,N'}(a) \in y$.

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[1]Lemmas 4.1 and 4.2 are also true by the same arguments when coherent adequate sets are replaced by symmetric systems in the sense of [1].

Case 2: $M' \cong N$. Since $a \in M' \cap N$, $\sigma_{M',N}(a) = a$. By the definition of y, $\sigma_{M,N'}(b) \in y$. But $\sigma_{M,N'}(b) = \sigma_{N,N'}(\sigma_{M',N}(\sigma_{M,M'}(b))) = \sigma_{N,N'}(\sigma_{M',N}(a)) = \sigma_{N,N'}(a)$. So $\sigma_{N,N'}(a) \in y$.

Case 3: $N < M'$. Fix $M_0 \in A$ isomorphic to M' such that $N \in M_0$. Let $N^* := \sigma_{M_0,M}(N)$, which is in A. Then $\sigma_{N^*,N} = \sigma_{M,M_0} \upharpoonright N^*$ by Lemma 1.10. Since $a \in M' \cap N$, $a \in M' \cap M_0$. By Lemma 1.13, $\sigma_{M_0,M}(a) = \sigma_{M',M}(a) = b$. In particular, since $a \in N$, $b = \sigma_{M_0,M}(a) \in \sigma_{M_0,M}(N) = N^*$. By the definition of y, $\sigma_{N^*,N'}(b) \in y$. But $\sigma_{N^*,N'}(b) = \sigma_{N,N'}(\sigma_{N^*,N}(b)) = \sigma_{N,N'}(\sigma_{M,M_0}(b)) = \sigma_{N,N'}(a)$. So $\sigma_{N,N'}(a) \in y$. $\qquad\square$

The next lemma analyzes closure in the context of amalgamation over countable models.

Lemma 4.2. *Let A be a coherent adequate set, $N \in A$, and suppose that B is a coherent adequate set with $A \cap N \subseteq B \subseteq N$. Let x and y be subsets of $H(\lambda)$ such that (x, A) is closed, (y, B) is closed, and $x \cap N \subseteq y \subseteq N$. Define C by*

$$\{M \in A : N \leq M\} \cup \{\sigma_{N,N'}(K) : N' \in A, \ N \cong N', \ K \in B\}.$$

Define z by

$$x \cup \{\sigma_{N,N'}(a) : N' \in A, \ N \cong N', \ a \in y\}.$$

Then the pair (z, C) is closed, $x \cup y \subseteq z$, and $z \cap N = y$.

Proof. Assume that K and M are in C and are isomorphic, and a is in $z \cap K$. We will show that $\sigma_{K,M}(a) \in z$.

Case 1: $N \leq K$. Then $N \leq M$. So both K and M are in A. If $a \in x$, then we are done since (x, A) is closed. So assume that $a = \sigma_{N,N'}(a_0)$ for some N' in A isomorphic to N and $a_0 \in y$.

Subcase 1a: K and M are isomorphic to N. Since $a \in K \cap N'$, $\sigma_{N',K}(a) = a$ by Lemma 1.13. By the definition of z, $\sigma_{N,M}(a_0) \in z$. But $\sigma_{N,M}(a_0) = \sigma_{K,M}(\sigma_{N',K}((\sigma_{N,N'}(a_0)))) = \sigma_{K,M}(\sigma_{N',K}(a)) = \sigma_{K,M}(a)$. So $\sigma_{K,M}(a) \in z$.

Subcase 1b: $N < K$. Since A is adequate, fix $J \in A$ isomorphic to K such that $N' \in J$. Let $N'' := \sigma_{J,K}(N')$ and $N''' := \sigma_{K,M}(N'')$. Then N'' and N''' are in A and are isomorphic to N. Moreover $\sigma_{N',N''} = \sigma_{J,K} \upharpoonright N'$ and $\sigma_{N'',N'''} = \sigma_{K,M} \upharpoonright N''$. By definition of z, $a_1 := \sigma_{N,N'''}(a_0)$ is in z. Since a is in $N' \cap K$, a is in $J \cap K$, so $\sigma_{J,K}(a) = a$. Hence $a_1 = \sigma_{N,N'''}(a_0) = \sigma_{N'',N'''}(\sigma_{N',N''}(\sigma_{N,N'}(a_0))) = \sigma_{N'',N'''}(\sigma_{N',N''}(a)) = \sigma_{K,M}(\sigma_{J,K}(a)) = \sigma_{K,M}(a)$. So $\sigma_{K,M}(a) \in z$.

Case 2: $K = \sigma_{N,N^*}(L)$ for some $N^* \in A$ isomorphic to N and $L \in B$. Since K and M are isomorphic, $M \cap \omega_1 = K \cap \omega_1 < N \cap \omega_1$. So also $M = \sigma_{N,N'}(P)$ for some $N' \in A$ isomorphic to N and $P \in B$. Then $L \cap \omega_1 = P \cap \omega_1$. Since B is coherent adequate, L and P are isomorphic.

Subcase 2a: $a = \sigma_{N,N''}(a_0)$ for some $N'' \in A$ isomorphic to N and $a_0 \in y$. Since $a \in K = \sigma_{N,N^*}(L)$, $a \in N'' \cap N^*$. By Lemma 1.13, $\sigma_{N^*,N}(a) = \sigma_{N'',N}(a) = a_0$. So $\sigma_{K,L}(a) = \sigma_{N^*,N}(a) = a_0$. It follows that $\sigma_{K,M}(a) = \sigma_{P,M}(\sigma_{L,P}(\sigma_{K,L}(a))) = \sigma_{P,M}(\sigma_{L,P}(a_0)) = \sigma_{N,N'}(\sigma_{L,P}(a_0))$. Since (y, B) is closed, $\sigma_{L,P}(a_0)$ is in y. So by the definition of z, $\sigma_{N,N'}(\sigma_{L,P}(a_0))$ is in z. That is, $\sigma_{K,M}(a)$ is in z.

Subcase 2b: a is in x. Since $a \in K = \sigma_{N,N^*}(L)$, it follows that $a \in N^*$. As (x, A) is closed, $a_0 := \sigma_{N^*,N}(a)$ is in $x \cap N$ and hence in y. But now $a = \sigma_{N,N^*}(a_0)$, N^* is in A, N^* is isomorphic to N, and $a_0 \in y$. So we are done by subcase 2a.

Now let us prove that $x \cup y \subseteq z$ and $z \cap N = y$. The set x is a subset of z by definition. And if $a \in y$, then $a = \sigma_{N,N}(a)$ is in z. So $x \cup y \subseteq z$. The set y is a subset of N by definition, and we just showed that y is a subset of z. So $y \subseteq z \cap N$. On the other hand, suppose that $a \in z \cap N$. If $a \in x$, then $a \in x \cap N$ and hence $a \in y$. Otherwise $a = \sigma_{N,N'}(a_0)$, where $N' \in A$ is isomorphic to N and $a_0 \in y$. But then $a \in N' \cap N$, so $\sigma_{N',N}(a) = a$. Hence $a = a_0$, so $a \in y$. $\qquad\square$

5. Amalgamating over Countable Models

In this section we prove an amalgamation result for countable models. The difference between this proposition and the analogous result from [5] is that here we provide an analysis of remainder points.

The next lemma considers a special case of the general result.

Lemma 5.1. *Let N, N', N^* be isomorphic in $\mathcal{X}$ such that $\{N, N', N^*\}$ is coherent adequate. Assume that K and L are in $N \cap \mathcal{X}$ and $\{K, L\}$ is adequate. Let $M := \sigma_{N,N'}(K)$ and $P := \sigma_{N,N^*}(L)$. Then $\{M, P\}$ is adequate. Moreover, if ζ is in $R_P(M) \cap \beta_{N',N^*}$, then $\sigma_{N',N}(\zeta)$ is in $R_L(K)$.*

Proof. Let $\sigma := \sigma_{N',N} \restriction N' \cap N^*$. By Lemma 1.13, σ is equal to $\sigma_{N^*,N} \restriction N' \cap N^*$.

Note that $\beta_{M,P} \leq \beta_{N',N^*}$, since $M \in N'$ and $P \in N^*$. Therefore $M \cap \beta_{M,P}$ is a countable subset of β_{N',N^*}. So $M \cap \beta_{M,P}$ is in $N' \cap P_{\omega_1}(\beta_{N',N^*})$. As N' and N^* are strongly isomorphic, $M \cap \beta_{M,N} \in N^*$ by Lemma 1.18. For the same reason, $P \cap \beta_{M,P}$ is in N'. Since $\beta_{M,P}$ is definable from $M \cap \beta_{M,P}$ as $\min(\Lambda \setminus \sup(M \cap \beta_{M,P}))$, we have that $\beta_{M,P} \in N' \cap N^*$. Also $\sigma(M \cap \beta_{M,P}) = \sigma_{N',N}(M) \cap \sigma(\beta_{M,P}) = K \cap \sigma(\beta_{M,P})$, and similarly $\sigma(P \cap \beta_{M,P}) = L \cap \sigma(\beta_{M,P})$.

We claim that $\sigma(\beta_{M,P}) \leq \beta_{K,L}$. Since $\beta_{M,P} \in \Lambda_M$, $\sigma(\beta_{M,P}) \in \sigma_{N',N}(\Lambda_M) = \Lambda_K$. Similarly, $\sigma(\beta_{M,P}) \in \Lambda_L$. Since $\beta_{K,L} = \max(\Lambda_K \cap \Lambda_L)$, $\sigma(\beta_{M,P}) \leq \beta_{K,L}$.

Let us show that $\{M, P\}$ is adequate. We will use the fact that $\{K, L\}$ is adequate and consider three cases.

Case 1: $K < L$. Then $K \cap \beta_{K,L} \in L$. Since $\sigma(\beta_{M,P}) \leq \beta_{K,L}$, $K \cap \sigma(\beta_{M,P}) \in L$. But $\sigma(M \cap \beta_{M,P}) = K \cap \sigma(\beta_{M,P})$ and $\sigma = \sigma_{N^*,N} \restriction N' \cap N^*$. Applying σ_{N,N^*}, we get that $\sigma^{-1}(K \cap \sigma(\beta_{M,N})) \in \sigma_{N,N^*}(L)$, that is $M \cap \beta_{M,P} \in P$.

Case 2: $L < K$. This case follows by a symmetric argument.

Case 3: $K \sim L$. As $\sigma(\beta_{M,P}) \leq \beta_{K,L}$, it follows that $K \cap \sigma(\beta_{M,P}) = L \cap \sigma(\beta_{M,P})$. As noted above, $\sigma(M \cap \beta_{M,P}) = K \cap \sigma(\beta_{M,P})$ and $\sigma(P \cap \beta_{M,P}) = L \cap \sigma(\beta_{M,P})$. So applying σ^{-1}, we get that $M \cap \beta_{M,P} = P \cap \beta_{M,P}$.

This completes the proof that $\{M, P\}$ is adequate. Now let ζ be in $R_P(M) \cap \beta_{N',N^*}$. We will show that $\zeta_0 := \sigma(\zeta)$ is in $R_L(K)$. Since $\zeta \in M$, applying $\sigma_{N',N}$, it follows that $\zeta_0 \in K$. Note that since $\beta_{M,P} \leq \zeta$, $\sigma(\beta_{M,P}) \leq \zeta_0$. So $\zeta_0 \in K \setminus \sigma(\beta_{M,P})$.

Case A: $M \leq P$. Then $K \leq L$. We claim that in this case, $\sigma(\beta_{M,P}) = \beta_{K,L}$. We already know that $\sigma(\beta_{M,P}) \leq \beta_{K,L}$. Suppose for a contradiction that $\sigma(\beta_{M,P}) <$

$\beta_{K,L}$. By Lemma 1.2, $K \cap [\sigma(\beta_{M,P}), \beta_{K,L})$ is nonempty. So letting $\tau_0 = \min(K \setminus \sigma(\beta_{M,P}))$, $\tau_0 < \beta_{K,L}$. Since $\sigma(\zeta) = \zeta_0 \in K \setminus \sigma(\beta_{M,P})$, $\tau_0 \le \zeta_0$ by the minimality of τ_0. So $\tau := \sigma_{N,N'}(\tau_0) \le \zeta$. Also since $\tau_0 \in K$, $\tau \in M$. And because $\sigma(\beta_{M,P}) \le \tau_0$, $\beta_{M,P} \le \tau$. So τ is in $M \setminus \beta_{M,P}$.

Since $K \le L$, $\tau_0 \in K \cap \beta_{K,L}$ implies that $\tau_0 \in L$. So $\tau_0 \in K \cap L$. As $\tau \le \zeta$ and $\zeta < \beta_{N',N^*}$, $\tau < \beta_{N',N^*}$. So $\tau = \sigma_{N,N'}(\tau_0) = \sigma_{N,N^*}(\tau_0)$ by Lemma 1.13. Since $\tau_0 \in L$, $\tau = \sigma_{N,N^*}(\tau_0) \in P$. So $\tau \in (M \cap P \cap \omega_2) \setminus \beta_{M,P}$, which is impossible. This contradiction shows that $\sigma(\beta_{M,P}) = \beta_{K,L}$.

Now we show that $\zeta_0 \in R_L(K)$. If $\zeta = \min(M \setminus \beta_{M,P})$, then applying $\sigma_{N',N}$ we get that $\zeta_0 = \min(K \setminus \sigma(\beta_{M,P})) = \min(K \setminus \beta_{K,L})$. So $\zeta_0 \in R_L(K)$. Now assume that there is $\gamma \in P \setminus \beta_{M,P}$ such that $\zeta = \min(M \setminus \gamma)$. Since $\zeta < \beta_{N',N^*}$, $\gamma < \beta_{N',N^*}$. Let $\gamma_0 := \sigma(\gamma)$. Applying $\sigma_{N',N}$, $\zeta_0 = \min(K \setminus \gamma_0)$. As $\sigma(\beta_{M,P}) = \beta_{K,L}$ and $\beta_{M,P} \le \gamma$, $\beta_{K,L} \le \gamma_0$. Also $\gamma \in P$ implies that $\gamma_0 = \sigma_{N^*,N}(\gamma) \in \sigma_{N^*,N}(P) = L$. So $\gamma_0 \in L \setminus \beta_{K,L}$. Therefore $\zeta_0 \in R_L(K)$.

Case B: $P < M$. Then $L < K$. As $\zeta \in R_P(M)$, there is $\gamma \in P \setminus \beta_{M,P}$ such that $\zeta = \min(M \setminus \gamma)$. Since $\gamma < \zeta < \beta_{N',N^*}$, $\gamma \in P \cap \beta_{N',N^*}$. Applying $\sigma_{N^*,N}$, $\gamma_0 := \sigma(\gamma)$ is in $\sigma_{N^*,N}(P) = L$. Since $\gamma \in P \setminus \beta_{M,P}$, γ is not in M. Applying $\sigma_{N',N}$, γ_0 is not in $\sigma_{N',N}(M) = K$. So $\gamma_0 \in L \setminus K$. Since $L < K$, this implies that $\beta_{K,L} \le \gamma_0$. Therefore $\gamma_0 \in L \setminus \beta_{K,L}$. Since $\zeta = \min(M \setminus \gamma)$, $\zeta_0 = \min(K \setminus \gamma_0)$. Hence $\zeta_0 \in R_L(K)$. $\qquad\square$

Proposition 5.2. *Let A be a coherent adequate set and $N \in A$. Suppose that B is a coherent adequate set and $A \cap N \subseteq B \subseteq N$. Let C be the set*

$$\{M \in A : N \le M\} \cup \{\sigma_{N,N'}(K) : N' \in A, \ N \cong N', \ K \in B\}.$$

Then C is a coherent adequate set, $A \cup B \subseteq C$, and $C \cap N = B$. Moreover, R_C is a subset of

$$R_A \cup \{\min(K \setminus \zeta) : K \in C, \ \zeta \in R_A\} \cup \{\sigma_{N,N'}(\tau) : N' \in A, \ N \cong N', \ \tau \in R_B\}.$$

Proof. Let M and P be in C such that $M \cap \omega_1 \le P \cap \omega_1$. We will show that $\{M, P\}$ is adequate and that the remainder points of M and P are as required.

Case 1: $N \cap \omega_1 \le M \cap \omega_1$. Then $N \cap \omega_1 \le M \cap \omega_1 \le P \cap \omega_1$. Hence M and P are both in A. So obviously $\{M, P\}$ is adequate and $R_M(P) \cup R_P(M) \subseteq R_A$.

Case 2: $M \cap \omega_1 < N \cap \omega_1 \le P \cap \omega_1$. Then clearly $P \in A$, $N \le P$, and $M = \sigma_{N,N'}(K)$ for some $N' \in A$ isomorphic to N and $K \in B$. Since $N \le P$ and N and N' are isomorphic, $N' \le P$. And as $M \in N'$, $\beta_{M,P} \le \beta_{N',P}$. So $M \cap \beta_{M,P}$ is in $N' \cap P_{\omega_1}(\beta_{N',P})$. As $N' \le P$, $M \cap \beta_{M,P} \in P$ by Lemma 1.19. So $M < P$.

Let $\zeta \in R_M(P)$ be given, and we will show that $\zeta \in R_A$. Since $M < P$, there is $\gamma \in M \setminus \beta_{M,P}$ such that $\zeta = \min(P \setminus \gamma)$. As $\beta_{M,P} \le \gamma$, γ is not in P. Since $N' \le P$ and $\gamma \in N' \setminus P$, $\beta_{N',P} \le \gamma$. So $\gamma \in N' \setminus \beta_{N',P}$. Therefore ζ is in $R_{N'}(P)$ and hence in R_A.

Now let $\zeta \in R_P(M)$ be given. Since $M < P$, there is $\gamma \in P \setminus \beta_{M,P}$ such that $\zeta = \min(M \setminus \gamma)$. As ζ is in $N' \setminus P$ and $N' \le P$, $\beta_{N',P} \le \zeta$. If $\gamma < \beta_{N',P}$, then $\zeta = \min(M \setminus \beta_{N',P})$. Let $\tau := \min(N' \setminus \beta_{N',P})$. Then $\tau \in R_A$. Since $M \subseteq N'$, $\zeta = \min(M \setminus \tau)$. Thus ζ is as required. Now assume that $\beta_{N',P} \le \gamma$. Then γ is in $P \setminus \beta_{N',P}$. Hence $\pi := \min(N' \setminus \gamma)$ is in R_A. But then $\zeta = \min(M \setminus \pi)$, so ζ is as required.

Case 3: $M \cap \omega_1 \leq P \cap \omega_1 < N \cap \omega_1$. Then $M = \sigma_{N,N'}(K)$ and $P = \sigma_{N,N^*}(L)$ for some N' and N^* in A isomorphic to N and K and L in B. By Lemma 5.1, $\{M, P\}$ is adequate. Moreover, if ζ is in $R_P(M) \cap \beta_{N',N^*}$ then $\sigma_{N',N}(\zeta) \in R_B$, and if ζ is in $R_M(P) \cap \beta_{N',N^*}$ then $\sigma_{N^*,N}(\zeta) \in R_B$. In either case, ζ is as required.

Let $\zeta \in R_P(M) \setminus \beta_{N',N^*}$ be given. Since $M \in N'$ and $P \in N^*$, $\beta_{M,P} \leq \beta_{N',N^*}$. If $\zeta = \min(M \setminus \beta_{M,P})$ or if $\zeta = \min(M \setminus \gamma)$ for some $\gamma \in P \setminus \beta_{M,P}$ which is below β_{N',N^*}, then $\zeta = \min(M \setminus \beta_{N',N^*})$. So letting $\tau := \min(N' \setminus \beta_{N',N^*})$, $\tau \in R_A$ and clearly $\zeta = \min(M \setminus \tau)$. So ζ is as required. Otherwise there is $\gamma \in P \setminus \beta_{M,P}$ such that $\beta_{N',N^*} \leq \gamma$ and $\zeta = \min(M \setminus \gamma)$. Since $\gamma \in P$ and $P \in N^*$, $\gamma \in N^* \setminus \beta_{N',N^*}$. Let $\pi := \min(N' \setminus \gamma)$. Then $\pi \in R_A$ and clearly $\zeta = \min(M \setminus \pi)$, and we are done. The proof that the remainder points in $R_M(P)$ are as required follows by a symmetric argument, since we never used in this paragraph the assumption that $M \cap \omega_1 \leq P \cap \omega_1$.

Now we show that $A \cup B \subseteq C$. If $K \in B$, then $K = \sigma_{N,N}(K)$ is in C. Let $M \in A$. If $N \cap \omega_1 \leq M \cap \omega_1$, then $M \in C$. Otherwise $M \cap \omega_1 < N \cap \omega_1$. Since A is coherent, there exists $N' \in A$ isomorphic to N such that $M \in N'$. Let $K := \sigma_{N',N}(M)$. Then K is in $A \cap N$ and hence in B. So $\sigma_{N,N'}(K) = M$ is in C.

It remains to prove that C is coherent and $C \cap N = B$. We apply Lemma 4.2 in the case where $x = A$ and $y = B$. Then clearly the set z defined there is equal to C. Since A and B are coherent, the pairs (A, A) and (B, B) are closed. So by Lemma 4.2, (C, C) is closed and $C \cap N = B$. Therefore if M and M' are in C and are isomorphic, and $K \in M \cap C$, then $\sigma_{M,M'}(K) \in C$.

We prove the remaining properties in the definition of coherence. Suppose that M and P are in C and $M \cap \omega_1 = P \cap \omega_1$. If $N \leq M$, then M and P are both in A and hence are strongly isomorphic. Otherwise $M < N$ and $P < N$, which implies that $M = \sigma_{N,N'}(K)$ and $P = \sigma_{N,N^*}(L)$ for some N' and N^* in A isomorphic to N and K and L in B. Then $K \cap \omega_1 = M \cap \omega_1 = P \cap \omega_1 = L \cap \omega_1$. Since B is coherent, K and L are strongly isomorphic. By Lemma 1.14, M and P are strongly isomorphic.

Now assume that M and P are in C and $M < P$. We will show that there is P' in C isomorphic to P such that $M \in P'$. If $N \cap \omega_1 \leq M \cap \omega_1$, then M and P are both in A, and we are done since A is coherent. Suppose that $M \cap \omega_1 < N \cap \omega_1 \leq P \cap \omega_1$. Then $M = \sigma_{N,N'}(K)$ for some $N' \in A$ isomorphic to N and $K \in B$. If N' is isomorphic to P then we are done. Otherwise $N' < P$, so there is $P' \in A$ isomorphic to P with $N' \in P'$. Then $M \in P'$.

Finally, assume that $M = \sigma_{N,N'}(K)$ and $P = \sigma_{N,N^*}(L)$ for some K and L in B and N' and N^* in A isomorphic to N. Since $M < P$, $K < L$. As B is coherent, fix L' in B isomorphic to L such that $K \in L'$. Then $M = \sigma_{N,N'}(K) \in \sigma_{N,N'}(L')$ and $\sigma_{N,N'}(L') \in C$. Since L and L' are strongly isomorphic in N, $\sigma_{N,N'}(L')$ and $\sigma_{N,N^*}(L) = P$ are strongly isomorphic by Lemma 1.14. $\square$

6. Adding a square sequence

We review the forcing poset from [5] for adding a square sequence with finite conditions, and show that it is in the class of coherent adequate type forcing posets. As a consequence, this forcing poset preserves CH.

By a *triple* we mean a sequence $\langle \alpha, \gamma, \beta \rangle$, where $\alpha \in \Lambda$ and $\gamma < \beta < \alpha$. Given distinct triples $\langle \alpha, \gamma, \beta \rangle$ and $\langle \alpha', \gamma', \beta' \rangle$, we say that the triples are *nonoverlapping* if either $\alpha \neq \alpha'$, or $\alpha = \alpha'$ and $[\gamma, \beta) \cap [\gamma', \beta') = \emptyset$; otherwise they are *overlapping*.

Definition 6.1. *Let $\mathbb{P}$ be the forcing poset whose conditions are pairs (x, A) satisfying:*

- (1) *x is a finite set of nonoverlapping triples;*
- (2) *A is a finite coherent adequate set;*
- (3) *for all $M \in A$ and $\langle \alpha, \gamma, \beta \rangle \in x$ such that $\alpha \in M$, either γ and β are in M or $\sup(M \cap \alpha) < \gamma$;*
- (4) *if M and M' are isomorphic sets in A, then for any triple $\langle \alpha, \gamma, \beta \rangle \in M \cap x$, $\sigma_{M,M'}(\langle \alpha, \gamma, \beta \rangle) \in x$.*

Let $(y, B) \leq (x, A)$ if $x \subseteq y$ and $A \subseteq B$.

We proved in [5] that $\mathbb{P}$ is strongly proper, ω_2-c.c., and forces $\square_{\omega_1}$.

Proposition 6.2. *The forcing poset $\mathbb{P}$ is a coherent adequate type forcing poset.*

Proof. Let (x, A) be condition in $\mathbb{P}$ and we will verify requirements (I)–(V) in the definition of a coherent adequate type forcing. The proof of Proposition 4.3 in [5] shows that if (x, A) is a condition and $M \in A$, then (x, A) is strongly M-generic. So requirement (V) holds. Obviously x is a subset of $H(\omega_2)$, and by assumption A is a coherent adequate set. Thus requirements (I) and (II) are immediate.

(III) Suppose that $(y, B) \leq (x, A)$, N and N' are isomorphic sets in B, and $(x, A) \in N$. Properties (1)–(4) in the definition of $\mathbb{P}$ are all first order definable in $H(\omega_2)$, except the part of (2) which asserts membership in $\mathcal{X}$. Since $\sigma_{N,N'}$ is an isomorphism which preserves membership in $\mathcal{X}$ and N and N' are elementary substructures of $H(\omega_2)$, $\sigma_{M,N}((x, A))$ is in $\mathbb{P}$. Also $\sigma_{M,N}((x, A)) = (\sigma_{M,N}[x], \sigma_{M,N}[A])$. Since (y, B) is closed and B is coherent, $x \subseteq y$ and $A \subseteq B$ imply that $\sigma_{M,N}[x] \subseteq y$ and $\sigma_{M,N}[A] \subseteq B$. Consequently, $(y, B) \leq \sigma_{M,N}((x, a))$.

(IV) Suppose that $M_0, \ldots, M_n$ are isomorphic sets in $\mathcal{X}$ such that $\{M_0, \ldots, M_n\}$ is coherent adequate and $(x, A) \in M_0 \cap \cdots \cap M_n$. We claim that $(x, A \cup \{M_0, \ldots, M_n\})$ satisfies properties (1)–(4) in the definition of $\mathbb{P}$. By Lemma 1.17, $A \cup \{M_0, \ldots, M_n\}$ is coherent adequate so (2) is satisfied. (1) is immediate. For (3), any triple in x is a member of $M_0, \ldots, M_n$, so there is nothing to check. And for (4), if $i, j \leq n$ and $a \in x \cap M_i$, then $a \in M_i \cap M_j$ so $\sigma_{M_i, M_j}(a) = a$ is in x. $\qquad \square$

7. Adding a club preserving CH

A forcing poset for adding a club to a fat subset of ω_2 using adequate sets as side conditions appears in [6]. Friedman [3] asked whether it is possible to add a club to ω_2 with finite conditions while preserving CH. In this section we adapt a variation of the poset from [6] to solve this problem.

Fix a stationary set $S \subseteq \omega_2$ which is fat. So for every club set $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$. Without loss of generality we may assume that for all $\alpha \in S \cap \text{cof}(\omega_1)$, $S \cap \alpha$ contains a closed cofinal subset of α. For the assumption of S being fat implies that there is a stationary subset of S satisfying this property. Also note that $S \cap \text{cof}(\omega_1)$ is stationary, since S is fat.

The general framework of coherent adequate sets introduced in the first section involves the parameters λ and Y. For this application we let $\lambda := \omega_2$ and let Y

code S together with a well-ordering of $H(\omega_2)$. So $\mathcal{X}$ consists of countable elementary substructures of $H(\omega_2)$, and isomorphisms between members of $\mathcal{X}$ preserve membership in S.

Let $\mathcal{Y}$ denote the set of M in $\mathcal{X}$ such that for all $\alpha \in (M \cap S) \cup \{\omega_2\}$, $\sup(M \cap \alpha) \in S$. A straightforward argument using the properties of S shows that $\mathcal{Y}$ is stationary in $P_{\omega_1}(H(\omega_2))$. Also since isomorphisms between members of $\mathcal{X}$ preserve membership in S, it is easy to check that $\mathcal{Y}$ is closed under isomorphisms.

For an ordinal α and a set N such that $N \cap \omega_2$ is not a subset of α, let α_N denote $\min(N \setminus \alpha)$. Given pairs of ordinals $\langle \alpha, \alpha' \rangle$ and $\langle \gamma, \gamma' \rangle$, we say that the pairs *overlap* if $\alpha < \gamma \le \alpha'$ or $\gamma < \alpha \le \gamma'$; otherwise they are *nonoverlapping*.

Definition 7.1. *Let $\mathbb{P}$ be the forcing poset consisting of conditions of the form $p = (x_p, A_p)$ satisfying:*

(1) *x_p is a finite set of nonoverlapping pairs of the form $\langle \alpha, \alpha' \rangle$, where $\alpha \le \alpha' < \omega_2$ and $\alpha \in S$;*

(2) *A_p is a finite coherent adequate subset of $\mathcal{Y}$;*

(3) *if $\langle \alpha, \alpha' \rangle \in x_p$, $N \in A_p$, and $N \cap \omega_2 \nsubseteq \alpha$, then $N \cap [\alpha, \alpha'] \ne \emptyset$ implies that α and α' are in N, and $N \cap [\alpha, \alpha'] = \emptyset$ implies that $\langle \alpha_N, \alpha_N \rangle \in x_p$;*

(4) *for all ζ in R_{A_p}, $\langle \zeta, \zeta \rangle \in x$;*

(5) *if M and N are in A_p and are isomorphic, then for any $a \in M \cap x_p$, $\sigma_{M,N}(a) \in x_p$.*

Let $q \le p$ if $x_p \subseteq x_q$ and $A_p \subseteq A_q$.

Observe that if (x, A) is a condition and $N \in A$, then $(x \cap N, A \cap N)$ is also a condition. Also, if (x, A) is in $\mathbb{P}$ and B is a subset of A which is coherent adequate, then (x, B) is a condition.

Note that in requirement (3), if the pair is of the form $\langle \alpha, \alpha \rangle$, then the conclusion in either case is equivalent to requiring that $\langle \alpha_N, \alpha_N \rangle \in x$.

Let $\dot{C}_S$ be a $\mathbb{P}$-name such that $\mathbb{P}$ forces

$$\dot{C}_S = \{\alpha : \exists p \in \dot{G} \ \exists \alpha' \ \langle \alpha, \alpha' \rangle \in x_p\}.$$

Clearly $\dot{C}_S$ is forced to be a subset of S. We will show that $\mathbb{P}$ is an $(S, \mathcal{Y})$ coherent adequate type forcing and $\mathbb{P}$ forces that $\dot{C}_S$ is club in ω_2.

Proposition 7.2. *Let $q = (x, A)$ be a condition and assume that $N \in A$. Then q is strongly N-generic.*

Proof. Fix a set D which is dense in the poset $N \cap \mathbb{P}$, and we will show that D is predense below q. Let $r \le q$ be given. Then $r \restriction N := (x_r \cap N, A_r \cap N)$ is a condition in N. Since D is dense, fix w in D below $r \restriction N$. We will prove that r and w are compatible.

Let C denote the set

$$\{M \in A_r : N \le M\} \cup \{\sigma_{N,N'}(K) : N' \in A_r, \ N \cong N', \ K \in A_w\}.$$

Let z denote the set

$$x_r \cup \{\sigma_{N,N'}(a) : N' \in A_r, \ N \cong N', \ a \in x_w\}.$$

Let $s := (z, C)$. We will show that s is a condition and $s \le r, w$.

By Proposition 5.2, C is a finite coherent adequate set, $A_r \cup A_w \subseteq C$, and $C \cap N = A_w$. Also C is a subset of $\mathcal{Y}$ since $\mathcal{Y}$ is closed under isomorphisms. So s satisfies requirement (2) in the definition of $\mathbb{P}$. By Lemma 4.2, the pair (z, C)

is closed, $x_r \cup x_w \subseteq z$, and $z \cap N = x_w$. Since (z, C) is closed, if M and M' are isomorphic sets in C, then for any $a \in M \cap z$, $\sigma_{M,M'}(a) \in z$. Thus s satisfies requirement (5) in the definition of $\mathbb{P}$. Since $x_r \cup x_w \subseteq z$ and $A_r \cup A_w \subseteq C$, it follows that if s is a condition, then $s \leq r, w$.

It remains to show that s satisfies requirements (1), (3), and (4) in the definition of $\mathbb{P}$. Regarding (1), it is easy to see that z consists of pairs $\langle \alpha, \alpha' \rangle$ where $\alpha \leq \alpha' < \omega_2$ and $\alpha \in S$, since this is true of pairs in x_r and x_w and these properties are preserved under isomorphisms. The proof that z consists of nonoverlapping pairs will use requirement (3), so we verify (3) first.

(3) Let $\langle \alpha, \alpha' \rangle \in z$ and $M \in C$ be given, and assume that $M \nsubseteq \alpha$. We will show that $M \cap [\alpha, \alpha'] \neq \emptyset$ implies that α and α' are in M, and $M \cap [\alpha, \alpha'] = \emptyset$ implies that $\langle \alpha_M, \alpha_M \rangle \in z$.

Case 1: $N \leq M$ and $\langle \alpha, \alpha' \rangle \in x_r$. Then M is in A_r. So we are done since r is a condition.

Case 2: $N \leq M$ and $\langle \alpha, \alpha' \rangle = \sigma_{N,N'}(\langle \alpha_0, \alpha_0' \rangle)$ for some $\langle \alpha_0, \alpha_0' \rangle \in x_w$ and $N' \in A_r$ isomorphic to N. Since $N \leq M$, $N' \leq M$. If $\alpha' < \beta_{M,N'}$, then $N' \cap \beta_{M,N'} \subseteq M$ implies that α and α' are in M. So assume that $\beta_{M,N'} \leq \alpha'$.

We claim that $R_M(N') \cap (\alpha, \alpha'] = \emptyset$. Otherwise assume that τ is in this intersection. Since r is a condition, $\langle \tau, \tau \rangle$ is in $x_r \cap N'$. Let $\tau_0 := \sigma_{N',N}(\tau)$. Then $\langle \tau_0, \tau_0 \rangle$ is in $x_r \cap N$ and hence in x_w. Since $\sigma_{N',N}$ is order preserving, $\alpha_0 < \tau_0 \leq \alpha_0'$, which contradicts that w is a condition.

If $\alpha < \beta_{M,N'} \leq \alpha'$, then $\min(N' \setminus \beta_{M,N'})$ is in $R_M(N') \cap (\alpha, \alpha']$, contradicting the claim. Therefore $\beta_{M,N'} \leq \alpha$. If $\xi \in M \cap [\alpha, \alpha']$, then $\xi \in M \setminus \beta_{M,N'}$. So $\min(N' \setminus \xi)$ is in $R_M(N') \cap (\alpha, \alpha']$, contradicting the claim. So $M \cap [\alpha, \alpha'] = \emptyset$. Since $\alpha \in N' \setminus \beta_{M,N'}$, $\alpha_M = \min(M \setminus \alpha)$ is in $R_{N'}(M)$. Hence $\langle \alpha_M, \alpha_M \rangle \in x_r$.

Case 3: $M = \sigma_{N,N'}(K)$ for some $K \in A_w$ and N' in A_r isomorphic to N and $N' \cap [\alpha, \alpha'] \neq \emptyset$. Then α and α' are in N' by cases 1 and 2. Let $\alpha_0 := \sigma_{N',N}(\alpha)$ and $\alpha_0' := \sigma_{N',N}(\alpha')$. Since s satisfies requirement (5), $\langle \alpha_0, \alpha_0' \rangle$ is in $z \cap N$. Since $z \cap N = x_w$, $\langle \alpha_0, \alpha_0' \rangle \in x_w$.

Assume that $M \cap [\alpha, \alpha'] \neq \emptyset$. Applying $\sigma_{N',N}$, $K \cap [\alpha_0, \alpha_0'] \neq \emptyset$. Since w is a condition, α_0 and α_0' are in K. Therefore their images under $\sigma_{N,N'}$, namely α and α', are in $\sigma_{N,N'}(K) = M$.

Now assume that $M \cap [\alpha, \alpha'] = \emptyset$. Applying $\sigma_{N',N}$, $K \cap [\alpha_0, \alpha_0'] = \emptyset$. Also $\sigma_{N',N}(\alpha_M) = \sigma_{N',N}(\min(M \setminus \alpha)) = \min(K \setminus \alpha_0) = (\alpha_0)_K$. Since w is a condition, $\langle (\alpha_0)_K, (\alpha_0)_K \rangle \in x_w$. Therefore $\sigma_{N,N'}(\langle (\alpha_0)_K, (\alpha_0)_K \rangle) = \langle \alpha_M, \alpha_M \rangle$ is in z.

Case 4: $M = \sigma_{N,N'}(K)$ for some $K \in A_w$ and N' in A_r isomorphic to N and $N' \cap [\alpha, \alpha'] = \emptyset$. As $M \in N'$, clearly $M \cap [\alpha, \alpha'] = \emptyset$. Since α_M exists, $\tau := \min(N' \setminus \alpha)$ exists. As $N \leq N'$, $\langle \tau, \tau \rangle \in z$ by Cases 1 and 2. If $\tau = \alpha_M$ then we are done. Otherwise $\tau < \alpha_M$ and $\alpha_M = \min(M \setminus \tau)$. Let $\tau_0 = \sigma_{N',N}(\tau)$. Since s satisfies requirement (5), $\langle \tau_0, \tau_0 \rangle$ is in $z \cap N = x_w$. Also $\min(K \setminus \tau_0)$ is equal to $\sigma_{N',N}(\min(M \setminus \tau))$, which is $\sigma_{N',N}(\alpha_M)$. As w is a condition, $\langle \sigma_{N',N}(\alpha_M), \sigma_{N',N}(\alpha_M) \rangle$ is in x_w. So the image of this pair under $\sigma_{N,N'}$, namely $\langle \alpha_M, \alpha_M \rangle$, is in z.

Now we show that z consists of nonoverlapping pairs. Suppose that $\langle \alpha, \alpha' \rangle$ and $\langle \gamma, \gamma' \rangle$ are in z, and we will prove that it is not the case that $\alpha < \gamma \leq \alpha'$. If both

pairs are in x_r then we are done since r is a condition, so assume not. Suppose for a contradiction that $\alpha < \gamma \le \alpha'$.

Case 1: $\langle \alpha, \alpha' \rangle = \sigma_{N,N'}(\langle \alpha_0, \alpha_0' \rangle)$ for some $\langle \alpha_0, \alpha_0' \rangle \in x_w$ and N' in A_r isomorphic to N, and $N' \cap [\gamma, \gamma'] = \emptyset$. Let $\tau := \min(N' \setminus \gamma)$, which exists because $\alpha' \in N'$. Then $\alpha < \tau \le \alpha'$. Since s satisfies requirement (3), $\langle \tau, \tau \rangle$ is in $z \cap N'$. Let $\tau_0 := \sigma_{N',N}(\tau)$. Since s satisfies requirement (5), $\langle \tau_0, \tau_0 \rangle$ is in $z \cap N = x_w$. But then $\alpha_0 < \tau_0 \le \alpha_0'$, contradicting that w is a condition.

Case 2: $\langle \alpha, \alpha' \rangle = \sigma_{N,N'}(\langle \alpha_0, \alpha_0' \rangle)$ for some $\langle \alpha_0, \alpha_0' \rangle \in x_w$ and N' in A_r isomorphic to N, and $N' \cap [\gamma, \gamma'] \ne \emptyset$. Then by requirement (3), γ and γ' are in N'. Let $\gamma_0 = \sigma_{N',N}(\gamma)$ and $\gamma_0' = \sigma_{N',N}(\gamma')$. Since s satisfies requirement (5), $\langle \gamma_0, \gamma_0 \rangle$ is in $z \cap N = x_w$. But then $\alpha_0 < \gamma_0 \le \alpha_0'$, contradicting that w is a condition.

Case 3: $\langle \gamma, \gamma' \rangle = \sigma_{N,N'}(\langle \gamma_0, \gamma_0' \rangle)$ for some $\langle \gamma_0, \gamma_0' \rangle \in x_w$ and $N' \in A_r$ isomorphic to N. Since $\alpha < \gamma \le \alpha'$, $N' \cap [\alpha, \alpha'] \ne \emptyset$. Since s satisfies requirement (3), α and α' are in N'. Let $\alpha_0 := \sigma_{N',N}(\alpha)$ and $\alpha_0' = \sigma_{N',N}(\alpha')$. Since s satisfies requirement (5), $\langle \alpha_0, \alpha_0' \rangle$ is in $z \cap N = x_w$. But then $\alpha_0 < \gamma_0 \le \alpha_0'$, contradicting that w is a condition.

Since at least one of $\langle \alpha, \alpha' \rangle$ and $\langle \gamma, \gamma' \rangle$ is not in x_r, these cases cover all possibilities.

(4) Let $\zeta \in R_C$ be given, and we will show that $\langle \zeta, \zeta \rangle \in z$. By Proposition 5.2, ζ is in the set

$$R_{A_r} \cup \{\min(K \setminus \xi) : K \in C, \ \xi \in R_{A_r}\} \cup \{\sigma_{N,N'}(\tau) : N' \in A, \ N \cong N', \ \tau \in R_{A_w}\}.$$

Let $\zeta \in R_C$ be given, and we will show that $\langle \zeta, \zeta \rangle \in z$.

Case 1: $\zeta \in R_{A_r}$. Since r is a condition, $\langle \zeta, \zeta \rangle \in x_r$.

Case 2: $\zeta = \sigma_{N,N'}(\tau)$ for some $\tau \in R_{A_w}$ and $N' \in A_r$ isomorphic to N. Since w is a condition, $\langle \tau, \tau \rangle \in x_w$. So $\sigma_{N,N'}(\langle \tau, \tau \rangle) = \langle \zeta, \zeta \rangle$ is in z by definition.

Case 3: For some $K \in C$ and $\xi \in R_{A_r}$, $\zeta = \min(K \setminus \xi)$. Then $\langle \xi, \xi \rangle \in x_r$. First assume that $K \in A_r$. Then since r is a condition, $\langle \zeta, \zeta \rangle \in x_r$.

Secondly assume that $K = \sigma_{N,N'}(L)$ for some $L \in A_w$ and $N' \in A_r$ isomorphic to N. Let $\tau := \min(N' \setminus \xi)$. Then $\zeta = \min(K \setminus \tau)$. Since r is a condition, $\langle \tau, \tau \rangle \in x_r$. Let $\tau_0 := \sigma_{N',N}(\tau)$. Then $\langle \tau_0, \tau_0 \rangle \in x_r \cap N = x_w$. Since $\zeta = \min(K \setminus \tau)$, applying $\sigma_{N',N}$ we get that $\zeta_0 := \sigma_{N',N}(\zeta) = \min(L \setminus \tau_0)$. Since w is a condition, $\langle \zeta_0, \zeta_0 \rangle$ is in x_w. Therefore $\sigma_{N,N'}(\langle \zeta_0, \zeta_0 \rangle) = \langle \zeta, \zeta \rangle$ is in z. $\square$

Proposition 7.3. *The forcing poset $\mathbb{P}$ is an $(S, \mathcal{Y})$ coherent adequate type forcing.*

Proof. Let (x, A) be a condition in $\mathbb{P}$, and we will show that (x, A) satisfies properties (I)–(V) in the definition of an $(S, \mathcal{Y})$ coherent adequate type forcing.

(I) is immediate, and (V) follows from Proposition 7.2. (II) By the definition of $\mathbb{P}$, A is a finite coherent adequate subset of $\mathcal{Y}$. If M and N are in A and $\zeta \in R_M(N)$, then $\langle \zeta, \zeta \rangle \in x$. This implies that $\zeta \in S$. Therefore A is (S) adequate.

(III) Let $(y, B) \le (x, A)$, and assume that N and N' are isomorphic in B with $(x, A) \in N$. Let $\sigma := \sigma_{N,N'}$. We will show that $\sigma((x, A))$ is in $\mathbb{P}$ and $(y, B) \le \sigma((x, A))$. All the properties of being in $\mathbb{P}$ described in Definition 7.1 are first order

definable in $H(\omega_2)$ without parameters, except for the requirements on membership in S and $\mathcal{Y}$. Therefore as N and N' are elementary in $H(\omega_2)$ and σ preserves membership in S and $\mathcal{Y}$, (x,A) being in $\mathbb{P}$ implies that $\sigma((x,A))$ is in $\mathbb{P}$. Also $\sigma((x,A)) = (\sigma[x], \sigma[A])$, and since (y,B) is a condition with $x \subseteq y$ and $A \subseteq B$, $\sigma[x] \subseteq y$ and $\sigma[A] \subseteq B$. So $(y,B) \leq \sigma((x,A))$.

(IV) Suppose that $M_0, \ldots, M_n$ are isomorphic sets in $\mathcal{Y}$ such that $\{M_0, \ldots, M_n\}$ is (S) coherent adequate and $(x,A) \in M_0 \cap \cdots \cap M_n$. Let $C = A \cup \{M_0, \ldots, M_n\}$. By Lemma 1.17, C is coherent adequate.

Let M and N be in C. If M is in $\{M_0, \ldots, M_n\}$ and N is in A, then $N \in M$. Hence $R_N(M)$ and $R_M(N)$ are empty. It follows that $R_C = R_A \cup \{R_{M_i}(M_j) : i,j \leq n\}$. In particular, $R_C \subseteq S$. So C is (S) coherent adequate.

Define
$$z = y \cup \{\sigma_{K,L}(a) : K, L \in C, \; K \cong L, \; a \in y \cap K\},$$
where $y = x \cup \{\langle \zeta, \zeta \rangle : \zeta \in R_C \setminus R_A\}$. By Lemma 4.1, (z, C) is closed. We will show that (z, C) is condition. Then clearly $(z, C) \leq (x, A)$ and $M_0, \ldots, M_k \in C$, which finishes the proof.

First we claim that z is equal to
$$z' := x \cup \{\sigma_{M_i, M_j}(\langle \zeta, \zeta \rangle) : i, j \leq n, \; \zeta \in (R_C \cap M_i) \setminus R_A\}.$$
Obviously $z' \subseteq z$.

For the other direction, first let us prove that $y \subseteq z'$. By definition, $x \subseteq z'$. Now consider $\zeta \in R_C \setminus R_A$. Then $\zeta \in R_{M_i}(M_j)$ for some $i, j \leq n$. Hence $\langle \zeta, \zeta \rangle = \sigma_{M_j, M_j}(\langle \zeta, \zeta \rangle)$ is in z'.

Secondly assume that $b = \sigma_{K,L}(a)$ where $K, L \in C$, $K \cong L$, and $a \in y \cap K$. Then either $K, L \in A$ or $K, L \in \{M_0, \ldots, M_n\}$. In the former case, a being in K implies that a is not in $R_C \setminus R_A$, since every member of $R_C \setminus R_A$ lies above every ordinal in $K \cap \omega_2$. Hence a is in x. So $b \in x$ since (x, A) is a condition.

Assume that $K, L \in \{M_0, \ldots, M_n\}$. If $a \in x$ then $a \in K \cap L$, so $b = \sigma_{K,L}(a) = a$. So $b \in y$, and hence $b \in z'$. Otherwise $a = \langle \zeta, \zeta \rangle$ where ζ is in $R_C \setminus R_A$, and then by definition b is in z'.

We have shown that $z = z'$. Note that if $\langle \alpha, \alpha \rangle$ is in $z \setminus x$, then for all $K \in A$ and $\langle \gamma, \gamma' \rangle$ in x, $\sup(K \cap \omega_2)$ and γ' are below α. For suppose $\langle \alpha, \alpha \rangle = \sigma_{M_i, M_j}(\langle \zeta, \zeta \rangle)$ for some $\zeta \in (R_C \cap M_i) \setminus R_A$. Then $\zeta \in R_{M_k}(M_l)$ for some $k, l \leq n$. So $\beta_{M_k, M_l} \leq \zeta$. Since $(x, A) \in M_k \cap M_l$, any ordinals in ω_2 definable from (x, A) are in β_{M_k, M_l} and hence are below ζ. If $\sigma_{M_i, M_j}(\zeta) = \zeta$ then we are done. Otherwise $\beta_{M_i, M_j} \leq \zeta$, so clearly $\beta_{M_i, M_j} \leq \sigma_{M_i, M_j}(\zeta)$. And as $(x, A) \in M_i \cap M_j$, again any ordinals definable from (x, A) are below $\sigma_{M_i, M_j}(\zeta)$.

We now verify that (z, C) satisfies properties (1)–(5) in the definition of $\mathbb{P}$. We already noted that C is a coherent adequate subset of $\mathcal{Y}$ and (z, C) is closed. Hence properties (2) and (5) holds. Also (4) is immediate.

(1) Since isomorphisms preserve membership in S, easily z consists of pairs of the form $\langle \alpha, \alpha' \rangle$ where $\alpha \leq \alpha' < \omega_2$ and $\alpha \in S$. Pairs which lie in x are nonoverlapping, and pairs lying in $z \setminus x$ are nonoverlapping because the first and second components in such a pair are equal. Consider $\langle \alpha, \alpha' \rangle \in x$ and $\sigma_{M_i, M_j}(\zeta)$ where $i, j \leq n$ and $\zeta \in (R_C \cap M_i) \setminus R_A$. By the comment above, $\alpha' < \sigma_{M_i, M_j}(\zeta)$. Hence $\langle \alpha, \alpha' \rangle$ and $\sigma_{M_i, M_j}(\langle \zeta, \zeta \rangle)$ do not overlap.

(3) Let $\langle \alpha, \alpha' \rangle \in z$ and $K \in C$ with $K \cap \omega_2 \nsubseteq \alpha$. If $\langle \alpha, \alpha' \rangle \in x$ and $K \in A$, then we are done. If $\langle \alpha, \alpha' \rangle \in x$ and $K \in \{M_0, \ldots, M_n\}$, then α and α' are in K.

Suppose that $\langle \alpha, \alpha' \rangle$ is equal to $\sigma_{M_i, M_j}(\langle \zeta, \zeta \rangle)$ for some $i, j \leq n$ and $\zeta \in (R_C \cap M_i) \setminus R_A$. Then $\alpha = \alpha'$. If $K \in A$, then by the comment above, $\sup(K \cap \omega_2) < \alpha$, contradicting our assumption. Therefore $K \in \{M_0, \ldots, M_n\}$. If $\alpha \in K$ then we are done, so assume not. Then $\alpha \in M_j \setminus K$. Since $K \cap \beta_{K, M_j} = M_j \cap \beta_{K, M_j}$, $\beta_{K, M_j} \leq \alpha$. So $\alpha \in M_j \setminus \beta_{K, M_j}$. It follows that $\alpha_K = \min(K \setminus \alpha)$ is in $R_{M_j}(K)$. By definition of z, $\langle \alpha_K, \alpha_K \rangle$ is in z. $\qquad \square$

It remains to show that $\mathbb{P}$ forces that $\dot{C}_S$ is a club. For unboundedness, given a condition p and an ordinal γ, choose β in S larger than all ordinals appearing in pairs of p and all suprema of models appearing in p intersected with ω_2. Then $(x_p \cup \{\langle \beta, \beta \rangle\}, A_p)$ is a condition which forces that $\dot{C}_S$ is not a subset of γ.

The proof of the closure of $\dot{C}_S$ is similar to the argument from [6], except that we have the new problem of needing to close under isomorphisms when adding something to a condition. This problem is dealt with by the next lemma.

Lemma 7.4. *Let (x, A) be a condition and let $\langle \alpha, \alpha' \rangle$ be a pair such that $\alpha \leq \alpha' < \omega_2$ and $\alpha \in S$. Assume:*

   (1) *$\langle \alpha, \alpha' \rangle$ does not overlap any pair in x;*
   (2) *if $N \in A$ and $N \cap \omega_2 \nsubseteq \alpha$, then $N \cap [\alpha, \alpha'] \neq \emptyset$ implies that α and α' are in N, and $N \cap [\alpha, \alpha'] = \emptyset$ implies that $\langle \alpha_N, \alpha_N \rangle \in x_p$.*

Let z be the set

$$x \cup \{\langle \alpha, \alpha' \rangle\} \cup \{\sigma_{N, N'}(\langle \alpha, \alpha' \rangle) : N, N' \in A_r, \ N \cong N', \ \langle \alpha, \alpha' \rangle \in N\}.$$

Then (z, A) is a condition below (x, A) and $\langle \alpha, \alpha' \rangle \in z$.

Proof. We will prove that (z, A) is a condition. Then it is clear that $(z, A) \leq (x, A)$ and $\langle \alpha, \alpha' \rangle \in x$. Properties (2) and (4) in the definition of $\mathbb{P}$ are immediate.

(5) Let $y := x \cup \{\langle \alpha, \alpha' \rangle\}$. Since (x, A) is closed, it is easy to see that z is equal to

$$y \cup \{\sigma_{N, N'}(a) : N, N' \in A, \ N \cong N', \ a \in y \cap N\}.$$

By Lemma 4.1, (z, A) is closed.

(3) Consider a pair $\langle \alpha_1, \alpha_1' \rangle = \sigma_{N, N'}(\langle \alpha, \alpha' \rangle)$, where N and N' are isomorphic in A and $\langle \alpha, \alpha' \rangle \in N$. Let $M \in A$ be given such that $M \cap \omega_2 \nsubseteq \alpha_1$.

We claim that $R_A \cap (\alpha_1, \alpha_1'] = \emptyset$. Suppose for a contradiction that ζ is in this intersection. Then $\langle \zeta, \zeta \rangle \in x$. So $\sigma_{N', N}(\langle \zeta, \zeta \rangle) \in x$. But this last pair overlaps with $\langle \alpha, \alpha' \rangle$, which contradicts our assumptions.

Case 1: $M \cap [\alpha_1, \alpha_1'] \neq \emptyset$. We will show that α_1 and α_1' are in M. Suppose that there is $\theta \in M \cap [\alpha_1, \alpha_1']$ such that $\beta_{M, N'} \leq \theta$. Let $\zeta := \min(N' \setminus \theta)$. Then ζ is in R_A and $\alpha_1 < \zeta \leq \alpha_1'$, contradicting the claim above. Therefore any ordinal in the nonempty intersection $M \cap [\alpha_1, \alpha_1']$ is strictly below $\beta_{M, N'}$. In particular, $\alpha_1 < \beta_{M, N'}$.

Assume that $N' \leq M$. If $\beta_{M, N'} \leq \alpha_1'$, then $\min(N' \setminus \beta_{M, N'})$ is in $R_A \cap (\alpha_1, \alpha_1']$, contradicting the claim above. So $\alpha_1' < \beta_{M, N'}$. Then α_1 and α_1' are in $N' \cap \beta_{M, N'}$ and hence in M.

Now assume that $M < N'$. Fix N^* in A isomorphic to N' such that $M \in N^*$. Since $M \in N^*$, $\beta_{M, N'} \leq \beta_{N^*, N'}$. In particular, $\alpha_1 < \beta_{N^*, N'}$. If $\beta_{N^*, N'} \leq \alpha_1'$, then $\min(N' \setminus \beta_{N^*, N'})$ is in $R_A \cap (\alpha_1, \alpha_1']$, contradicting the claim. Therefore $\alpha_1' < \beta_{N^*, N'}$. It follows that α_1 and α_1' are in N^*.

By Lemma 1.13, $\langle \alpha, \alpha' \rangle = \sigma_{N',N}(\langle \alpha_1, \alpha_1' \rangle)$ is equal to $\sigma_{N^*,N}(\langle \alpha_1, \alpha_1' \rangle)$. Since $M \cap [\alpha_1, \alpha_1'] \neq \emptyset$, applying $\sigma_{N^*,N}$ we get that $\sigma_{N^*,N}(M) \cap [\alpha, \alpha'] \neq \emptyset$. As $\sigma_{N^*,N}(M) \in A$, our assumptions imply that α and α' are in $\sigma_{N^*,N}(M)$. Applying σ_{N,N^*}, we get that α_1 and α_1' are in M.

Case 2: $M \cap [\alpha_1, \alpha_1'] = \emptyset$. Let $\beta := \min(M \setminus \alpha_1)$, and we will show that $\langle \beta, \beta \rangle \in z$. Note that β equals $\min(M \setminus \alpha_1')$. So if $\beta_{M,N'} \leq \alpha_1'$, then $\beta \in R_A$ and hence $\langle \beta, \beta \rangle \in x$. So assume that $\alpha_1' < \beta_{M,N'}$. Since M does not contain α_1, we must have that $M < N'$. If $\beta_{M,N'} \leq \beta$, then $\beta = \min(M \setminus \beta_{M,N'})$. So $\beta \in R_A$ and again $\langle \beta, \beta \rangle \in x$. Assume that $\beta < \beta_{M,N'}$. Since $M < N'$, it follows that $\beta \in M \cap N'$.

Fix N^* in A which is isomorphic to N' with $M \in N^*$. Since $\beta \in N' \cap N^*$, $\beta < \beta_{N',N^*}$. So α_1 and α_1' are in N^*. By Lemma 1.13, $\sigma_{N^*,N}(\langle \alpha_1, \alpha_1' \rangle) = \sigma_{N',N}(\langle \alpha_1, \alpha_1' \rangle) = \langle \alpha, \alpha' \rangle$. Since $M \cap [\alpha_1, \alpha_1'] = \emptyset$, applying $\sigma_{N^*,N}$ we get that $\sigma_{N^*,N}(M) \cap [\alpha, \alpha'] = \emptyset$. Let $\pi := \min(\sigma_{N^*,N}(M) \setminus \alpha)$. As $\sigma_{N^*,N}(M) \in A$, our assumptions imply that $\langle \pi, \pi \rangle \in x$. But $\beta = \min(M \setminus \alpha_1)$, so applying $\sigma_{N^*,N}$ we get that $\sigma_{N^*,N}(\beta) = \min(\sigma_{N^*,N}(M) \setminus \alpha) = \pi$. Hence $\langle \sigma_{N^*,N}(\beta), \sigma_{N^*,N}(\beta) \rangle$ is in x. Since (x, A) is closed, $\langle \beta, \beta \rangle \in x$.

(1) It is immediate that the pairs of z are of the correct form. We will show that they are nonoverlapping. This is true by assumption for all pairs in $x \cup \{\langle \alpha, \alpha' \rangle\}$.

Case 1: Suppose that $\langle \gamma, \gamma' \rangle \in x$ and $\langle \alpha_1, \alpha_1' \rangle = \sigma_{N,N'}(\langle \alpha, \alpha' \rangle)$, where N and N' are isomorphic in A and $\langle \alpha, \alpha' \rangle \in N$. First assume that $N' \cap [\gamma, \gamma'] \neq \emptyset$. Then γ and γ' are in N'. Hence any overlap between $\langle \alpha_1, \alpha_1' \rangle$ and $\langle \gamma, \gamma' \rangle$ translates to an overlap between $\langle \alpha, \alpha' \rangle$ and $\sigma_{N',N}(\langle \gamma, \gamma' \rangle)$. As $\sigma_{N',N}(\langle \gamma, \gamma' \rangle) \in x$, this would contradict our assumptions.

Assume that $N' \cap [\gamma, \gamma'] = \emptyset$. Then the only possible overlap between $\langle \gamma, \gamma' \rangle$ and $\langle \alpha_1, \alpha_1' \rangle$ would be if $\alpha_1 < \gamma \leq \gamma' < \alpha_1'$. Letting $\zeta := \min(N' \setminus \gamma)$, we have that $\langle \zeta, \zeta \rangle \in x$ and $\alpha_1 < \zeta \leq \alpha_1'$. But now we get a contradiction exactly as in the previous paragraph.

Case 2: Suppose that $\langle \alpha_1, \alpha_1' \rangle = \sigma_{N,N'}(\langle \alpha, \alpha' \rangle)$ and $\langle \alpha_2, \alpha_2' \rangle = \sigma_{M,M'}(\langle \alpha, \alpha' \rangle)$, where N, N', M, M' are in A, $N \cong N'$, and $M \cong M'$. Suppose for a contradiction that $\alpha_1 < \alpha_2 \leq \alpha_1'$. Then $M' \cap [\alpha_1, \alpha_1'] \neq \emptyset$. By property (3) of (z, C), this implies that α_1 and α_1' are in M'. Since these ordinals are also in N', $\alpha_1' < \beta_{M',N'}$.

First assume that M and N are isomorphic. As α and α' are in $M \cap N$, $\sigma_{M,N}(\alpha) = \alpha$ and $\sigma_{M,N}(\alpha') = \alpha'$. Since $\alpha_1 < \alpha_2 \leq \alpha_1' < \beta_{M',N'}$, these three ordinals are in $M' \cap N'$, so $\sigma_{M',N'}$ fixes them. In particular, we have that $\alpha_1 = \sigma_{N,N'}(\alpha) = \sigma_{M',N'}(\sigma_{M,M'}(\sigma_{N,M}(\alpha))) = \sigma_{M',N'}(\sigma_{M,M'}(\alpha)) = \sigma_{M',N'}(\alpha_2) = \alpha_2$. But this contradicts that $\alpha_1 < \alpha_2$.

Assume that $M < N$. Fix N^* in A isomorphic to N with $M \in N^*$. Then α and α' are in $N \cap N^*$. So $\sigma_{N,N'}$ and $\sigma_{N^*,N'}$ agree on α and α' by Lemma 1.13. Let $L := \sigma_{N^*,N'}(M)$, which is in A and satisfies that $\sigma_{M,L} = \sigma_{N^*,N'} \restriction M$. In particular, $\sigma_{M,L}(\langle \alpha, \alpha' \rangle) = \sigma_{N^*,N'}(\langle \alpha, \alpha' \rangle) = \sigma_{N,N'}(\langle \alpha, \alpha' \rangle) = \langle \alpha_1, \alpha_1' \rangle$. So $\langle \alpha_1, \alpha_1' \rangle = \sigma_{M,L}(\langle \alpha, \alpha' \rangle)$ and $\langle \alpha_2, \alpha_2' \rangle = \sigma_{M,M'}(\langle \alpha, \alpha' \rangle)$, and thus we have reduced the current situation to the case considered in the previous paragraph. A symmetric argument works when $N < M$. $\qquad\square$

Note the following consequence of the lemma: if (x, A) is a condition and $\langle \beta, \beta' \rangle \in x$, then there is $(y, A) \le (x, A)$ such that $\langle \beta, \beta \rangle \in y$. Also the lemma implies that for all p, there is $q \le p$ such that for all $\langle \beta, \beta' \rangle \in x_q$, $\langle \beta, \beta \rangle \in x_q$. For the process of adding $\langle \beta, \beta \rangle$ to p described in the lemma adds only pairs of the form $\langle \gamma, \gamma \rangle$, where for some γ', $\langle \gamma, \gamma' \rangle \in x_p$. So repeating finitely many times we can close the condition under this requirement.

Proposition 7.5. *The forcing poset $\mathbb{P}$ forces that $\dot{C}_S$ is a club.*

Proof. Suppose that p forces that α is a limit point of $\dot{C}_S$. We will find a condition below p which forces that $\alpha \in \dot{C}_S$. If $\langle \alpha, \alpha' \rangle \in x_p$ for some α', then p forces that $\alpha \in \dot{C}_S$ and we are done. So assume not. Then for all $\langle \xi, \xi' \rangle$ in x_p, either $\xi \le \xi' < \alpha$ or $\alpha < \xi \le \xi'$.

Let $A_0 := \{ K \in A_p : \sup(K \cap \omega_2) < \alpha \}$, $A_1 := \{ K \in A_p : \sup(K \cap \alpha) < \alpha, K \cap \omega_2 \not\subseteq \alpha \}$, and $A_2 := \{ K \in A_p : \sup(K \cap \alpha) = \alpha \}$. Note that for all M and N in A_2, α is a limit point of both M and N and therefore $\alpha < \beta_{M,N}$.

Case 1: $\mathrm{cf}(\alpha) = \omega_1$. Extending p if necessary, we may assume that there is $M \in A_p$ such $\alpha \in M$. Then $M \cap \alpha$ is bounded below α. Since p forces that α is a limit point of $\dot{C}_S$, we can fix $q \le p$ such that for some γ and γ', $\sup(M \cap \alpha) < \gamma \le \gamma' < \alpha$ and $\langle \gamma, \gamma' \rangle \in x_q$. Then $M \cap [\gamma, \gamma'] = \emptyset$ and $\min(M \setminus \gamma) = \alpha$. So $\langle \alpha, \alpha \rangle$ is in x_q since q is a condition.

Case 2: $\mathrm{cf}(\alpha) = \omega$. Since p forces that α is a limit point of $\dot{C}_S$, fix $t \le p$ satisfying:

    (a) there is γ and γ' satisfying that $\gamma \le \gamma' < \alpha$, $\langle \gamma, \gamma' \rangle \in x_t$, and for all $K \in A_0 \cup A_1$, $\sup(K \cap \alpha) < \gamma$;

    (b) γ is the largest such ordinal;

    (c) for all pairs $\langle \beta, \beta' \rangle$ in x_t, $\langle \beta, \beta \rangle$ is in t.

Let $q := (x_t, A_p)$. Then q is a condition and $q \le p$. If $\langle \alpha, \alpha' \rangle \in x_q$ for some α' then we are done, so assume not. It follows that $\langle \alpha, \alpha \rangle$ does not overlap any pair in x_q since α is forced to be a limit point of $\dot{C}_S$. Also by the maximality of γ, the pair $\langle \gamma, \alpha \rangle$ does not overlap any pair in x_q. For all $K \in A_1$, $K \cap [\gamma, \gamma'] = \emptyset$, and clearly $\alpha_K = \gamma_K$. Hence $\langle \alpha_K, \alpha_K \rangle \in x_q$. In particular, for all $K \in A_1$, $\alpha < \alpha_K$.

Subcase 2a: $A_2 = \emptyset$. We will use Lemma 7.4 to show we can add $\langle \gamma, \alpha \rangle$ to q, contradicting that α is forced to be a limit point of $\dot{C}_S$. By the choice of γ, $\gamma \in S$ and $\langle \gamma, \alpha \rangle$ does not overlap any pair in x_q. By the case assumption, if $K \in A_p$ and $K \cap \omega_2 \not\subseteq \gamma$, then $K \in A_1$. By the choice of γ and the comments above, $K \cap [\gamma, \alpha] = \emptyset$ and $\langle \alpha_K, \alpha_K \rangle \in x_q$. By Lemma 7.4 there is an extension of q which contains $\langle \gamma, \alpha \rangle$.

Subcase 2b: $A_2 \ne \emptyset$ and there exists $M \in A_2$ such that $\sup(M \cap \omega_2) = \alpha$. We apply Lemma 7.4 to show that we can add $\langle \alpha, \alpha \rangle$. Note that $\alpha \in S$ since $M \in \mathcal{Y}$. And $\langle \alpha, \alpha \rangle$ does not overlap any pair in x_q as noted above. Let $N \in A_q$ be given such that $N \cap \omega_2 \not\subseteq \alpha$ and $\alpha \notin N$. Then $N \notin A_0$. If $N \in A_1$ then we already know that $\langle \alpha_N, \alpha_N \rangle \in x_q$. Suppose that $N \in A_2$. Then $\alpha < \beta_{M,N}$ as pointed out above. In particular, $\alpha = \sup(M \cap \beta_{M,N})$. Since $\alpha \notin N$, it is not the case that $M < N$. As α is a limit point of $N \cap \beta_{M,N}$ not in M, likewise N cannot be below

M. Therefore $M \sim N$. Since $\alpha_N \notin M$, $\beta_{M,N} \leq \alpha_N$. So $\alpha < \beta_{M,N} \leq \alpha_N$. It follows that $\alpha_N = \min(N \setminus \beta_{M,N})$. So $\alpha_N \in R_M(N)$. Therefore $\langle \alpha_N, \alpha_N \rangle \in x_q$.

Subcase 2c: $A_2 \neq \emptyset$ and for all $M \in A_2$, $M \cap \omega_2 \not\subseteq \alpha$. Let M be a member of A_2 satisfying that for all $K \in A_2$, (i) $M \cap \omega_1 \leq K \cap \omega_1$, and (ii) if $M \cap \omega_1 = K \cap \omega_1$ then $\alpha_M \leq \alpha_K$.

First assume that there is $\langle \beta, \beta' \rangle$ in x_q with $\alpha \leq \beta \leq \alpha_M$. Then $\beta_M = \alpha_M$. By our assumptions, β cannot equal α, so $\alpha < \beta$. By the choice of q, $\langle \beta, \beta \rangle \in x_q$. Therefore $\langle \alpha_M, \alpha_M \rangle \in x_q$. In particular, $\alpha_M \in S \cap M$. By the definition of $\mathcal{Y}$, $\sup(M \cap \alpha_M) = \alpha \in S$.

We apply Lemma 7.4 to show that we can add $\langle \alpha, \alpha \rangle$. The pair $\langle \alpha, \alpha \rangle$ does not overlap any pair in x_q. Let $N \in A_q$ be given such that $N \cap \omega_2 \not\subseteq \alpha$ and $\alpha \notin N$. We will show that $\langle \alpha_N, \alpha_N \rangle \in x_q$. Obviously $N \notin A_0$. And if $N \in A_1$, then we already know that $\langle \alpha_N, \alpha_N \rangle \in x_q$. Let $N \in A_2$. Then by (i), $M \leq N$. Recall that $\alpha < \beta_{M,N}$. Since $\alpha \notin N$ and α is a limit point of $M \cap \beta_{M,N}$, it is not possible that $M < N$. So $M \cap \omega_1 = N \cap \omega_1$. By (ii), $\alpha_M \leq \alpha_N$. But then $\alpha < \beta \leq \alpha_N$, so $\beta_N = \alpha_N$. Since q is a condition, $\langle \beta_N, \beta_N \rangle \in x_q$. So $\langle \alpha_N, \alpha_N \rangle \in x_q$.

Now assume that there is no such pair $\langle \beta, \beta' \rangle$ in x_q. In particular, $\langle \alpha_M, \alpha_M \rangle$ is not in x_q. We apply Lemma 7.4 to show that we can add $\langle \gamma, \alpha_M \rangle$ to q. This is a contradiction, since any extension of q forces that α is a limit point of $\dot{C}_S$.

Since $\langle \gamma, \gamma \rangle \in x_q$, it follows that $\langle \gamma_M, \gamma_M \rangle \in x_q$ since q is a condition. By the maximality of γ, $\gamma = \gamma_M$. So $\gamma \in M$. Also by the maximality of γ, $\langle \gamma, \alpha_M \rangle$ does not overlap any pair in x_q of ordinals below α. And if $\langle \beta, \beta' \rangle \in x_q$ with $\alpha \leq \beta$, then $\alpha_M < \beta$ by our case assumption. Hence $\langle \gamma, \alpha_M \rangle$ does not overlap any pair in x_q.

Now let $N \in A_q$ be given such that $N \cap \omega_2 \not\subseteq \alpha$. We will show that either γ and α_M are in N, or $N \cap [\gamma, \alpha_M] = \emptyset$ and $\langle \gamma_N, \gamma_N \rangle \in x_q$. Obviously $N \notin A_0$. Suppose that $N \in A_1$. Then $\langle \alpha_N, \alpha_N \rangle \in x_q$ as noted above. And by the choice of γ, $N \cap \alpha \subseteq \gamma$. So $\gamma_N = \alpha_N$. Hence $\langle \gamma_N, \gamma_N \rangle \in x_q$.

Suppose that $N \in A_2$. First assume that for all $K \in A_2$, $N \cap \omega_1 \leq K \cap \omega_1$. Then by (i) and (ii), $N \cap \omega_1 = M \cap \omega_1$ and $\alpha_M \leq \alpha_N$. Since $\alpha < \beta_{M,N}$ and $\gamma \in M \cap \alpha$, $\gamma \in N$. We claim $\alpha_M = \alpha_N$. If not, then $\beta_{M,N} \leq \alpha_M$. But then $\alpha < \beta_{M,N} \leq \alpha_M$ implies that $\alpha_M = \min(M \setminus \beta_{M,N})$. So $\alpha_M \in R_A$, and therefore $\langle \alpha_M, \alpha_M \rangle \in x_q$, which contradicts our case assumption. It follows that γ and α_M are both in N.

Now assume that $M < N$. Recall that $\alpha < \beta_{M,N}$. So if $\beta_{M,N} \leq \alpha_M$, then $\alpha_M = \min(M \setminus \beta_{M,N})$ and hence $\alpha_M \in R_N(M)$. This implies that $\langle \alpha_M, \alpha_M \rangle \in x_q$, which contradicts our case assumption. So $\alpha_M < \beta_{M,N}$. Since $M \cap \beta_{M,N} \subseteq N$, γ and α_M are in N. $\qquad\square$

8. Preserving cardinals larger than ω_2

In this final section we will prove a general amalgamation result for models of size ω_1. As a consequence, we show that the forcing poset consisting of finite coherent adequate sets in $H(\lambda)$ ordered by inclusion is ω_2-c.c.[2] Although we did not need this fact for the other results in this paper, we record it here for future applications.

Lemma 8.1. *Let M_0, M_1, K_0, and K_1 be in $\mathcal{X}$. Let $\beta \in \Lambda$. Assume that $M_0 \cap \beta = M_1 \cap \beta$, $K_0 \cap \beta = K_1 \cap \beta$, $\beta_{K_0,M_0} \leq \beta$, and $\beta_{K_1,M_1} \leq \beta$. Then $\beta_{K_0,M_0} = \beta_{K_1,M_1}$.*

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[2]If $2^{\omega_1} = \omega_2$ and $\lambda^{<\lambda} = \lambda$, then the chain condition follows immediately from the conjunction of Proposition 4.4 of [4] and Lemma 2.5 of [1]. A proof of Lemma 2.5 of [1] can be found in Lemma 3.9 of [2].

Proof. Clearly $K_0 \cap \beta = K_1 \cap \beta$ and $M_0 \cap \beta = M_1 \cap \beta$ imply that $\Lambda_{M_0} \cap (\beta + 1) = \Lambda_{M_1} \cap (\beta + 1)$ and $\Lambda_{K_0} \cap (\beta + 1) = \Lambda_{K_1} \cap (\beta + 1)$. Since β_{K_0,M_0} and β_{K_1,M_1} are both in $\beta + 1$, $\beta_{K_0,M_0} = \max(\Lambda_{K_0} \cap \Lambda_{M_0}) = \max(\Lambda_{K_0} \cap \Lambda_{K_1} \cap (\beta + 1)) = \max((\Lambda_{K_0} \cap (\beta+1)) \cap (\Lambda_{M_0} \cap (\beta+1))) = \max((\Lambda_{K_1} \cap (\beta+1)) \cap (\Lambda_{M_1} \cap (\beta+1))) = \max(\Lambda_{K_1} \cap \Lambda_{M_1} \cap (\beta + 1)) = \max(\Lambda_{K_1} \cap \Lambda_{M_1}) = \beta_{K_1,M_1}$. $\square$

Proposition 8.2. *Let A be a finite coherent adequate set. Let $\chi \geq \lambda$ be a regular cardinal. Assume that N^* is an elementary substructure of $(H(\chi), \in)$ of size ω_1 such that $\beta^* := N^* \cap \omega_2 \in \Lambda$. Let $\beta \in \beta^* \cap \Lambda$.*

Assume that there exists a map $M \mapsto M'$ from A into N^* satisfying:

(1) *$M \cong M'$, $M \cap \beta^* = M' \cap \beta$, and $M \cap N^* \subseteq M'$;*
(2) *$K \in M$ iff $K' \in M'$, and in that case, $\sigma_{M,M'}(K) = K'$;*
(3) *if $M \in N^*$ then $M = M'$;*
(4) *$A' := \{M' : M \in A\}$ is a coherent adequate set.*

Then $C := A \cup A'$ is a coherent adequate set. Moreover, if we let $r_A = \{\min(M \setminus \beta^*) : M \in A\}$ and $r_{A'} = \{\min(M' \setminus \beta) : M \in A\}$, then $R_C \subseteq R_A \cup R_{A'} \cup r_A \cup r_{A'}$.

Proof. For all $M \in A$, since $M \cap \beta^* = M' \cap \beta$, $\sigma_{M,M'} \upharpoonright \beta$ is the identity function. Also note that $M \cap \beta^* = M' \cap \beta$ obviously implies that $M \cap \beta^* \subseteq \beta$ and $M \cap \beta = M' \cap \beta$.

First we will prove that C is adequate and that the remainder points of C are as required. Since A and A' are each adequate, we only need to compare models in A with models in A'. So let K and M be in A and we will compare K and M'. Since $M' \in N^*$, $M' \cap \omega_2 \subseteq \beta^*$. So $\beta_{K,M'} \leq \beta^*$ by Lemma 1.7(1). If $\beta < \beta_{K,M'}$ then by Lemma 1.2, $K \cap [\beta, \beta_{K,M'})$ is nonempty. But then $K \cap [\beta, \beta^*)$ is nonempty, which is false. So $\beta_{K,M'} \leq \beta$.

We claim that either $\beta^* \leq \beta_{K,M}$ or $\beta_{K,M} = \beta_{K,M'}$. In particular, $\beta_{K,M'} \leq \beta_{K,M}$. For suppose that $\beta_{K,M} < \beta^*$. If $\beta < \beta_{K,M}$, then by Lemma 1.7(2), there is $\xi \in K \cap M \cap [\beta, \beta_{K,M})$. But then $\xi \in K \cap \beta^* \subseteq \beta$, which is a contradiction. So $\beta_{K,M} \leq \beta$. Now apply Lemma 8.1 letting $K_0 = K_1 = K$, $M_0 = M$, and $M_1 = M'$. Then obviously $K_0 \cap \beta = K_1 \cap \beta$, $M_0 \cap \beta = M_1 \cap \beta$, $\beta_{K_0,M_0} = \beta_{K,M} \leq \beta$, and $\beta_{K_1,M_1} = \beta_{K,M'} \leq \beta$. So by Lemma 8.1, $\beta_{K,M} = \beta_{K,M'}$.

Now we show that $\{K, M'\}$ is adequate.

Case 1: $K < M$. Then $K \cap \beta_{K,M} \in M$. Since $\beta_{K,M'} \leq \beta_{K,M}$, $K \cap \beta_{K,M'} \in M$. As $\beta_{K,M'} \leq \beta$ and $K \cap \beta = K' \cap \beta$, $K \cap \beta_{K,M'} = K' \cap \beta_{K,M'}$. Since $K' \in N^*$, $K \cap \beta_{K,M'} \in M \cap N^*$. But $M \cap N^* \subseteq M'$. So $K \cap \beta_{K,M'} \in M'$, and therefore $K < M'$.

Case 2: $K \sim M$. Since $\beta_{K,M'} \leq \beta_{K,M}$, $K \cap \beta_{K,M'} = M \cap \beta_{K,M'}$. As $\beta_{K,M'} \leq \beta$ and $M \cap \beta = M' \cap \beta$, $M \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$. So $K \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$.

Case 3: $M < K$. Then $M \cap \beta_{K,M} \in K$. Since $\beta_{K,M'} \leq \beta_{K,M}$, $M \cap \beta_{K,M'} \in K$. But $\beta_{K,M'} \leq \beta$ and $M \cap \beta = M' \cap \beta$. So $M \cap \beta_{K,M'} = M' \cap \beta_{K,M'}$. Therefore $M' \cap \beta_{K,M'} \in K$. Hence $M' < K$.

Now we show that the remainder points are as required.

Let $\zeta \in R_K(M')$ be given. First assume that $\zeta \geq \beta$. If $\zeta = \min(M' \setminus \beta_{K,M'})$, then since $\beta_{K,M'} \leq \beta$, $\zeta = \min(M' \setminus \zeta)$. So $\zeta \in r_{A'}$ and we are done. Otherwise

$\zeta = \min(M' \setminus \gamma)$ for some $\gamma \in K \setminus \beta_{K,M'}$. Since $M' \cap \omega_2 \subseteq \beta^*$, $\zeta < \beta^*$. As $K \cap \beta^* \subseteq \beta$, $\gamma < \beta$. So again $\zeta = \min(M' \setminus \beta)$ and $\zeta \in r_{A'}$.

Secondly, assume that $\zeta < \beta$. Then $\zeta \in M' \cap \beta = M \cap \beta$. Since $\beta_{K,M'} \leq \zeta$, ζ is not in K. Suppose that $M \leq K$. Since $\zeta \in M \setminus K$, $\beta_{K,M} \leq \zeta$. As $\zeta < \beta$, $\beta_{K,M} < \beta$. Therefore by the claim above, $\beta_{K,M} = \beta_{K,M'}$. If $\zeta = \min(M' \setminus \beta_{K,M'})$, then clearly $\zeta = \min(M \setminus \beta_{K,M})$ and $\zeta \in R_A$. Suppose that $\zeta = \min(M' \setminus \gamma)$ for some $\gamma \in K \setminus \beta_{K,M'}$. Then $\gamma \in K \setminus \beta_{K,M}$ and $\zeta = \min(M \setminus \gamma)$, so $\zeta \in R_A$.

Now assume that $K < M$. Then $\zeta = \min(M' \setminus \gamma)$ for some $\gamma \in K \setminus \beta_{K,M'}$. Since $\zeta < \beta$, $\gamma < \beta$. Hence γ is not in M since otherwise it would be in M'. So $\gamma \in K \setminus M$. Since $K < M$, this implies that $\beta_{K,M} \leq \gamma$. So $\beta_{K,M} = \beta_{K,M'}$ by the claim above. Therefore $\gamma \in K \setminus \beta_{K,M}$ and $\zeta = \min(M \setminus \gamma)$. Hence ζ is in R_A.

Now consider $\zeta \in R_{M'}(K)$. First assume that $\beta \leq \zeta$. Since $K \cap \beta^* \subseteq \beta$, $\beta^* \leq \zeta$. But $\beta_{K,M'}$ and any ordinal in $M' \cap \omega_2$ are below β^*. So clearly $\zeta = \min(K \setminus \beta^*)$ and $\zeta \in r_A$.

Now suppose that $\zeta < \beta$. First assume that $K \leq M$. Since $\zeta \notin M'$ and $M' \cap \beta = M \cap \beta$, $\zeta \notin M$. So $\zeta \in K \setminus M$. Since $K \leq M$, this implies that $\beta_{K,M} \leq \zeta$. Therefore $\beta_{K,M} < \beta$ and hence $\beta_{K,M} = \beta_{K,M'}$ by the claim above. If $\zeta = \min(K \setminus \beta_{K,M'})$, then $\zeta = \min(K \setminus \beta_{K,M})$ and therefore $\zeta \in R_A$. Otherwise $\zeta = \min(K \setminus \gamma)$ for some $\gamma \in M' \setminus \beta_{K,M'}$. Since $\zeta < \beta$, $\gamma < \beta$. So $\gamma \in M' \cap \beta = M \cap \beta$. Hence $\gamma \in M \setminus \beta_{K,M}$, so $\zeta \in R_A$.

Next assume that $M < K$. Then $\zeta = \min(K \setminus \gamma)$ for some $\gamma \in M' \setminus \beta_{K,M'}$. Since $\zeta < \beta$, $\gamma \in M' \cap \beta = M \cap \beta$. So $\gamma \in M$. As $\gamma \in M \setminus K$ and $M < K$, $\beta_{K,M} \leq \gamma < \beta$. By the claim above, $\beta_{K,M} = \beta_{K,M'}$. Hence $\gamma \in M \setminus \beta_{K,M}$, so $\zeta \in R_A$.

Next we will prove that C is coherent. We verify the conditions (1), (2), and (3) of Definition 1.15.

(1) Let K and M be in A and assume that $K \sim M'$. Then $K \cap \omega_1 = M' \cap \omega_1 = M \cap \omega_1$, so K and M are strongly isomorphic. Since M and M' are isomorphic, K and M' are isomorphic. To see that K and M' are strongly isomorphic, let $a \in K \cap M'$ be given. Since $a \in M'$ and $M' \in N^*$, $a \in K \cap N^*$. Since $K \cap N^* \subseteq K'$, $a \in K'$. So $a \in K \cap K'$, which implies that $\sigma_{K,K'}(a) = a$ since K and K' are strongly isomorphic by assumption. Also $a \in K' \cap M'$, so $\sigma_{K',M'}(a) = a$ since K' and M' are strongly isomorphic. So $\sigma_{K,M'}(a) = \sigma_{K',M'}(\sigma_{K,K'}(a)) = \sigma_{K',M'}(a) = a$.

(2) Let K and M be in A. First assume that $K < M'$, and we will find M^* in C isomorphic to M' such that $K \in M^*$. Since $K \cap \omega_1 < M' \cap \omega_1 = M \cap \omega_1$, $K < M$. By the coherence of A, there is M^* in A isomorphic to M such that $K \in M^*$. Then M^* is isomorphic to M' and we are done.

Now assume that $M' < K$, and we will find K^* in C isomorphic to K such that $M' \in K^*$. We have that $M' \cap \omega_1 < K \cap \omega_1 = K' \cap \omega_1$, so $M' < K'$. Since A' is coherent, there is K^* in A' isomorphic to K' such that $M' \in K^*$. Then K^* is isomorphic to K.

(3) Let M, N, and K be in C, where M and N are isomorphic and $K \in M$. We will prove that $\sigma_{M,N}(K) \in C$.

We claim that if M is in A then so is K. So assume that $K = L'$ for some $L \in A$. Since $L' \in M \cap N^*$ and $M \cap N^* \subseteq M'$, $L' \in M \cap M'$. Therefore $\sigma_{M,M'}(L') = L'$.

But $L' \in M'$ implies that $L \in M$ and $\sigma_{M,M'}(L) = L'$. Hence $L = L'$ and therefore $L' = K$ is in A.

Case 1: M and N are in A. Then $K \in A$, so we are done since A is coherent.

Case 2: M is in A and $N = P'$ for some $P \in A$. Then again K is in A. Let $Q = \sigma_{M,P}(K)$, which is in A since A is coherent. Then $\sigma_{P,P'}(Q) = Q'$ is in A'. And $\sigma_{M,P'}(K) = \sigma_{P,P'}(\sigma_{M,P}(K)) = \sigma_{P,P'}(Q) = Q'$ which is in A' and hence in C.

Case 3: $M = M_0'$ and $N = N_0'$ for some M_0 and N_0 in A. If $K = L'$ for some $L \in A$, then we are done since A' is coherent. Suppose that $K \in A$. Then $K \in M_0'$ and $M_0' \in N^*$, which imply that $K \in N^*$. So $K = K'$. Hence $K \in A'$ so $\sigma_{M,N}(K) \in A'$ since A' is coherent. $\qquad\square$

Theorem 8.3. *Assume CH. Then the forcing poset consisting of finite coherent adequate sets ordered by inclusion is ω_2-c.c.*

Proof. Let $\mathcal{A}$ be a maximal antichain, and we will prove that $|\mathcal{A}| \le \omega_1$. Fix a regular cardinal $\chi > \lambda$ such that $\mathbb{P}$ is in $H(\lambda)$. By CH, we can fix N^* an elementary substructure of $H(\chi)$ of size ω_1 such that Y, $\mathbb{P}$, and $\mathcal{A}$ are in N^* and $(N^*)^\omega \subseteq N^*$.

Since N^* has size ω_1, we will be done if we can show that $\mathcal{A} \subseteq N^*$. Let $\beta^* := N^* \cap \omega_2$. Note that β^* is in Λ. And by elementarity, $\Lambda \cap \beta^*$ is cofinal in β^*.

Let $A = \{M_0, \ldots, M_k\}$ be a condition in $\mathcal{A}$. Fix $\beta \in \Lambda \cap N^*$ large enough so that for all $M \in A$, $\sup(M \cap \beta^*) < \beta$. Let R be the relation where $R(i,j)$ holds if $M_i \in M_j$. Let d be the set of $i \le k$ such that $M_i \in N^*$.

For each $i \le k$ let $\mathfrak{M}_i$ denote the structure $(M_i, \in, Y \cap M_i)$, and let $\overline{\mathfrak{M}}_i$ denote its transitive collapse. Recall that σ_{M_i} is the transitive collapsing map of $\mathfrak{M}_i$. For each pair $\langle i, j \rangle$ in R, let $J_{i,j} := \sigma_{M_j}(M_i)$. For each $i \le k$ let S_i be the relation where $S_i(a,b)$ holds if $a \in M_i \cap N^*$ and $\sigma_{M_i}(a) = b$.

Since R and d are finite, they are in N^*. By CH, $H(\omega_1) \subseteq N^*$. So each transitive collapse $\overline{\mathfrak{M}}_i$ and each object $J_{i,j}$ is in N^*. As N^* is countably closed, each set $M_i \cap N^*$ is a member of N^*. So each relation S_i is in N^*.

The objects $M_0, \ldots, M_k$ witness in the model $H(\chi)$ that there exist $M_0', \ldots, M_k'$ satisfying:

(a) $\{M_0', \ldots, M_k'\}$ is a finite coherent adequate set in $\mathcal{A}$;
(b) for $i \le k$, $M_i' \cap \beta = M_i \cap \beta^*$, $M_i \cap N^* \subseteq M_i'$, and the transitive collapse of the structure $(M_i', \in, Y \cap M_i')$ is equal to $\overline{\mathfrak{M}}_i$;
(c) $R(i,j)$ iff $M_i' \in M_j'$, and in that case, $\sigma_{M_j'}(M_i') = J_{i,j}$;
(d) for $i \le k$, if $S_i(a,b)$ then $\sigma_{M_i'}(a) = b$;
(e) for $i \in d$, $M_i = M_i'$.

The parameters mentioned in the above statement are those described in the previous paragraph together with $\mathcal{A}$, β, $M_i \cap \beta^*$ for $i \le k$, and M_i for $i \in d$. So the parameters are all members of N^*. By the elementarity of N^*, fix $M_0', \ldots, M_k'$ in N^* satisfying the same statement.

Let us show that the map $M_i \mapsto M_i'$ for $i \le k$ satisfies the assumptions (1)–(4) of Proposition 8.2. By (a), $\{M_0', \ldots, M_k'\}$ is a coherent adequate set, so (4) holds. For (3), if $M_i \in N^*$, then $i \in d$, and so by (e), $M_i = M_i'$.

(1) Consider $i \le k$. By (b), $M_i \cap \beta^* = M_i' \cap \beta$ and $M_i \cap N^* \subseteq M_i'$. Also by (b), the structures $(M_i, \in, Y \cap M_i)$ and $(M_i', \in, Y \cap M_i')$ have the same transitive collapse and hence are isomorphic. To see that $M_i \cong M_i'$, let $a \in M_i \cap M_i'$. Then

$a \in M_i \cap N^*$. Let $b := \sigma_{M_i}(a)$. Then $S_i(a, b)$ holds. By (d), $\sigma_{M_i'}(a) = b$. Hence $\sigma_{M_i, M_i'}(a) = \sigma_{M_i'}^{-1}(\sigma_{M_i}(a)) = \sigma_{M_i'}^{-1}(b) = a$.

(2) By the definition of R and (c), $M_i \in M_j$ iff $R(i, j)$ iff $M_i' \in M_j'$. And in that case, $\sigma_{M_j'}(M_i') = J_{i,j}$. Now $J_{i,j} = \sigma_{M_j}(M_i)$ by definition. So $\sigma_{M_j, M_j'}(M_i) = \sigma_{M_j'}^{-1}(\sigma_{M_j}(M_i)) = \sigma_{M_j'}^{-1}(J_{i,j}) = M_i'$.

This completes the verification of the assumptions of Proposition 8.2. Let $C := A \cup A'$. Then C is a finite coherent adequate set, and obviously $C \le A, A'$. Hence A and A' are compatible. But A and A' are both in $\mathcal{A}$. Since $\mathcal{A}$ is an antichain, $A = A'$. Therefore $A \in N^*$. Thus we have shown that $\mathcal{A} \subseteq N^*$, completing the proof. $\qquad\square$

By a somewhat easier argument, if CH fails then the forcing poset consisting of finite coherent adequate sets ordered by inclusion satisfies the $(2^\omega)^+$-c.c. Hence by the material in Section 3, if $2^\omega < \lambda$ then this forcing poset collapses 2^ω to have size ω_1, forces CH, and preserves all cardinals above 2^ω.

This paper has introduced a new method for producing models of CH using finite forcing. This raises the question whether the method can be used to produce new independence results involving CH.

References

[1] D. Asperó and M.A. Mota. Separating club-guessing principles in the presence of fat forcing axioms. Submitted.

[2] D. Asperó and M.A. Mota. Forcing consequences of PFA together with the continuum large. *Trans. Amer. Math. Soc.*, 367(9):6103–6129, 2015.

[3] S.D. Friedman. Forcing with finite conditions. In *Set Theory: Centre de Recerca Matemàtica, Barcelona, 2003-2004, Trends in Mathematics*, pages 285–295. Birkhäuser Verlag, 2006.

[4] J. Krueger. Forcing with adequate sets of models as side conditions. Submitted.

[5] J. Krueger. Coherent adequate sets and forcing square. *Fund. Math.*, 224(3):279–300, 2014.

[6] J. Krueger. Adding a club with finite conditions, part II. *Arch. Math. Logic*, 54(1-2):161–172, 2015.

[7] W. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. *Trans. Amer. Math. Soc.*, 361(2):561601, 2009.

[8] T. Miyamoto. Forcing a quagmire via matrices of models. Preprint.

[9] I. Neeman. Forcing with sequences of models of two types. *Notre Dame J. Form. Log.*, 55(2):265–298, 2014.

[10] S. Todorčević. A note on the proper forcing axiom. In *Axiomatic set theory (Boulder, Colo., 1983)*, volume 31 of *Contemp. Math.*, pages 209–218. Amer. Math. Soc., Providence, RI, 1984.

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