SEPARATING CLUB–GUESSING PRINCIPLES IN THE PRESENCE OF FAT FORCING AXIOMS

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Abstract. We separate various weak forms of Club Guessing at ω_1 in the presence of $2^{\aleph_0}$ large, Martin's Axiom, and related forcing axioms.

We also answer a question of Abraham and Cummings concerning the consistency of the failure of a certain polychromatic Ramsey statement together with the continuum large.

All these models are generic extensions via finite support iterations with symmetric systems of structures as side conditions, possibly enhanced with ω–sequences of predicates, and in which the iterands are taken from a relatively small class of forcing notions.

We also prove that the natural forcing for adding a large symmetric system of structures (the first member in all our iterations) adds $\aleph_1$–many reals but preserves CH.

1. Introduction

One is sometimes faced with the problem of building a model of set theory satisfying the following two requirements.

(1) $2^{\aleph_0} > \aleph_2$ holds in the model.
(2) Some particular combinatorial principle P of the form "For all x there is some y such that $Q(x, y)$", where $Q(x, y)$ is sufficiently absolute, holds in the model. Furthermore, for each x there is a natural proper forcing adding a y such that $Q(x, y)$. Hence, P can be forced by means of a countable support iteration of proper forcings, but in the corresponding extension $2^{\aleph_0} \leq \aleph_2$ necessarily holds.

The method of iterated forcing with finite supports and symmetric systems of submodels as side conditions was developed in [6] in order to resolve the tension between (1) and (2) in various situations (see

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[6] and [7] for background information). Variants of this method have been subsequently investigated in [7] and [3].

One of the central themes of the present article is the construction of iterations as in [6] where the iterands are chosen from some relatively small class of posets (let us call these constructions of the first type). The other central theme is a new variation of the general method from [6] obtained from associating sequences of predicates to the submodels in the side conditions of the iteration (this gives rise to the constructions of the second type). The main focus in this article is the separation of club–guessing principles at ω_1 in the presence of forcing axioms for relatively small classes of posets but with respect to large collections of dense sets. The corresponding models are obtained as forcing extensions via constructions of either the first or the second type. Also, using a construction of the first type we answer a question of Abraham–Cummings in the context of polychromatic Ramsey theory ([1]).

The rest of the paper is structured as follows. In the next subsection we prove several implications and non-implications between weak forms of Club Guessing at ω_1, and present our main theorems (Theorems 1.15 and 1.16). In Section 2 we take a look at the forcing for adding a symmetric system of submodels by finite conditions. This forcing is either the first step or is subsumed in the first step in all our iterations. We show that this forcing adds $\aleph_1$–many reals but preserves CH. In Section 3 we prove Theorems 1.15 and 1.16. Finally, in Section 4 we deal with the Abraham–Cummings question. Most of our notation will be standard (see e.g. [12] or [15]) but we will also be introducing additional pieces of notation as we need them.

1.1. **Weak forms of Club Guessing.** A *ladder system* is a sequence $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$, where each C_δ is a club of δ of order type ω. *Club Guessing* (CG) is the well–known weakening of $\diamondsuit$ saying that there is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ which guesses clubs C of ω_1, in the sense that for every such C there is some δ such that a tail of C_δ is contained in C. In this subsection we focus our attention on certain weakenings of CG. The web of implications between these principles will be immediate. We will then point out how to prove several non–implications between these principles, with a focus on what can be obtained in the presence of forcing axioms for large families of dense sets. Finally we present our main separation theorems, to be proved in Section 3.[1]

[1]For further information on these (and other related) club–guessing principles, see for example [11], [10], and [17].

Kunen's Axiom (KA), also known as *Interval Hitting Principle* (see for example [10]), is the following statement first considered by Kunen: There is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ with the property that for every club $C \subseteq \omega_1$ there is some δ such that $[C_\delta(n), C_\delta(n+1)) \cap C \neq \emptyset$ for co-finitely many $n < \omega$ (where, here and throughout the paper, $X(\xi)$ denotes the ξ–th member of X if X is a set of ordinals).

$\mho$ (mho), first defined by Todorčević ([22]), says that there is a sequence of continuous colourings $g_\delta : \delta \longrightarrow \omega$, for $\delta \in \mathrm{Lim}(\omega_1)$, where δ and ω are both endowed with the order topology (so the topology on ω is the discrete topology), such that for every club $C \subseteq \omega_1$ there is some δ with $g_\delta^{-1}(\{n\}) \cap C \neq \emptyset$ for all $n < \omega$.

It is clear that CG implies KA and that KA implies $\mho$.

Weak Club Guessing (WCG), first defined by Shelah ([19]), says that there is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ such that every club of ω_1 has infinite intersection with some C_δ. Very Weak Club Guessing (VWCG), also first considered by Shelah, says that there is a set $\mathcal{X}$ of size $\aleph_1$ consisting of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with some member of $\mathcal{X}$.

One can weaken VWCG even further: Given a cardinal $\lambda \geq \aleph_1$, VWCG_λ says that there is a set $\mathcal{X}$ of size at most λ consisting of subsets of ω_1 of order type ω and such that every club of ω_1 has infinite intersection with some member of $\mathcal{X}$. $\neg\,\mathrm{VWCG}_\lambda$ is called $(*)^\omega_\lambda$ in [7] (Definition 1.10).

Obviously WCG implies VWCG and VWCG_λ implies VWCG_μ whenever $\aleph_1 \leq \lambda < \mu$.

By a result of Shelah, $\neg\,$WCG is compatible with CH ([20]; see also [19]). On the other hand, CH obviously implies VWCG. In fact we have the following.[2]

Fact 1.1. *(Hrušák)* $\mathfrak{b} \leq \lambda$ *implies* VWCG_λ.

Proof. Let $\lambda' \leq \lambda$ and let $(r_\alpha \mid \alpha < \lambda')$ be such that $\{r_\alpha\}_{\alpha<\lambda'}$ is an $<^*$–unbounded subset of ${}^\omega\omega$ (where $f <^* g$ means that $f(n) < g(n)$ for a tail of n). Let $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ be a ladder system and let $h_\delta : \omega \longrightarrow \delta$ be a bijection for each $\delta \in \mathrm{Lim}(\omega_1)$. For all $\alpha < \lambda'$ and $\delta \in \mathrm{Lim}(\omega_1)$ let

$$A^\alpha_\delta = \{C_\delta(n) \mid n < \omega\} \cup \bigcup\{(h_\delta``[0, r_\alpha(n))) \setminus C_\delta(n) \mid n < \omega\}$$

It is easy to check that for all α and δ, A^α_δ has order type ω and $\sup(A^\alpha_\delta) = \delta$. Given a club $C \subseteq \omega_1$, let $\delta \in C$ be a limit point of C

[2]Thanks to M. Hrušák for pointing this out in the case $\lambda = \aleph_1$ (the proof for general λ is the same).

and let $g_{C,\delta} : \omega \longrightarrow \omega$ be the function given by

$$g_{C,\delta}(n) = \min\{m \mid h_\delta(m) \in C \setminus C_\delta(n)\}$$

Now let $\alpha < \lambda'$ be such that $r_\alpha(n) > g_{C,\delta}(n)$ for infinitely many n. It then follows that $|A^\alpha_\delta \cap C| = \aleph_0$. $\square$

Let us consider the following notion of rank (see e.g. [5] and [6]).

Definition 1.2. Given a set X and an ordinal δ, we define the *Cantor–Bendixson rank of δ with respect to X*, $\mathrm{rank}(X, \delta)$, by specifying that

- $\mathrm{rank}(X, \delta) \geq 1$ if and only if δ is a limit point of ordinals in X.
- If $\mu > 1$, $\mathrm{rank}(X, \delta) \geq \mu$ if and only and for every $\eta < \mu$, δ is a limit of ordinals ϵ with $\mathrm{rank}(X, \epsilon) \geq \eta$.

Let us call an ordinal δ *perfect* if $\mathrm{rank}(\delta, \delta) = \delta$.

Given any uncountable cardinal κ, the set of perfect ordinals in κ form a club of order type κ. Hence, if $\kappa \geq \omega_1$ is a regular cardinal and $C \subseteq \kappa$ is a club, the set of perfect ordinals δ in κ which are fixed points of the enumerating function of C forms a club of κ.

The following fact will be useful at some point.

Fact 1.3. *Let A and A' be sets of ordinals, let δ and τ be ordinals, and suppose* $\mathrm{rank}(A, \delta) < \tau$ *and* $\mathrm{rank}(A', \delta) < \tau$. *Then* $\mathrm{rank}(A \cap A', \delta) < \tau$.

Given an ordinal τ, we will say that a set X of ordinals is *τ–thin* in case $\mathrm{rank}(X, \delta) < \tau$ for all ordinals δ. One can strengthen $\neg\,\mathrm{VWCG}_\lambda$ even further in the following way.

Definition 1.4. ([7]) Given ordinals τ and λ, $\tau < \omega_1$, $(\cdot)^\tau_\lambda$ is the following statement: For every sequence $(X_i)_{i<\lambda}$, if X_i is a τ–thin subset of ω_1 for all $i < \lambda$, then there is a club $C \subseteq \omega_1$ such that $|C \cap X_i| < \aleph_0$ for all i.

One can define the 'strong' form of the (weak) guessing principles we have been looking at by requiring that the relevant guessing occurs on a club of δ's. For example, *Strong Club Guessing* (*Strong* CG) is the statement that there is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ such that for every club $C \subseteq \omega_1$, $|C_\delta \setminus C| < \aleph_0$ for club–many $\delta \in \mathrm{Lim}(\omega_1)$. Similarly we can define strong KA, strong $\mho$, strong Weak Club Guessing, and so on. In particular, a strong $\mho$–sequence is a sequence of continuous colourings $g_\delta : \delta \longrightarrow \omega$, for $\delta \in \mathrm{Lim}(\omega_1)$, such that for every club $C \subseteq \omega_1$ there are club–many $\delta < \omega_1$ with $g_\delta^{-1}(\{n\}) \cap C \neq \emptyset$ for all $n < \omega$, and a strong WCG–sequence is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ such that for every club $C \subseteq \omega_1$ there are club–many δ such that $C_\delta \cap C$ is infinite.

Note the following facts, which will be used in Section 3. In the statement of these facts, and throughout the paper, we adopt the convention of denoting by δ_N the ordinal $N \cap \omega_1$ if N is a set such that $N \cap \omega_1$ is an ordinal. We will sometimes call δ_N *the height of* N. Of course, if N is countable and sufficiently correct, then δ_N exists and moreover $\delta_N \in C$ for every club $C \subseteq \omega_1$ in N. This is the main reason behind the following two results.

Fact 1.5. *(Folklore) If N is a countable elementary substructure of $H(\omega_2)$ containing a strong $\mho$–sequence $\langle g_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$, then for every club $C \subseteq \omega_1$ in N and every $n \in \omega$ there is some $\varepsilon \in C \cap \delta_N$ such that $g_{\delta_N}(\varepsilon) = n$.*

Fact 1.6. *(Folklore) If N is a countable elementary substructure of $H(\omega_2)$ containing a strong* WCG*–sequence $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$, then for every club $C \subseteq \omega_1$ in N, $C_{\delta_N} \cap C$ is infinite.*

Of course Strong P implies P for all these guessing principles P, but the reverse implications do not hold. Also, Strong P_1 implies Strong P_0 whenever P_1 implies P_0.[3] In particular, strong CG implies both strong $\mho$ and strong WCG. These strong guessing principles are consistent:

Fact 1.7. *(Folklore)* CH *+ Strong* CG *can always be forced.*

Proof. Assume CH and let $\mathcal{P}$ be the forcing for adding a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ by initial segments. In $V^{\mathcal{P}}$, let $\dot{\mathcal{Q}}$ be a countable support iteration shooting all relevant clubs so as to make $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ strongly club–guessing. It is easy to check that $\mathcal{P} * \dot{\mathcal{Q}}$ is a proper poset not adding reals and that after forcing with $\dot{\mathcal{Q}}$ over $\mathbf{V}^{\mathcal{P}}$, the generic $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ becomes strongly club–guessing. $\square$

We fix some notation regarding forcing axioms: Given a partial order $\mathcal{P}$ and a cardinal λ, $\mathrm{FA}(\mathcal{P})_\lambda$ means: For every collection $\{D_i \mid i < \lambda\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$. Given a class Γ of partial orders and a cardinal λ, $\mathrm{FA}(\Gamma)_\lambda$ means $\mathrm{FA}(\mathcal{P})_\lambda$ for every $\mathcal{P} \in \Gamma$.

PFA (i.e., $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P} \text{ proper}\})_{\aleph_1}$), and in fact BPFA, implies both $\neg$ VWCG and $\neg\mho$ (by a standard argument using a natural poset for adding, by finite approximations, a club destroying the relevant guessing sequence).[4] On the other hand, every club of ω_1 in every c.c.c.

[3] One should be careful: Even if $\diamondsuit$ implies CG, $\diamondsuit^+$ (which is a 'weakly strong' form of $\diamondsuit$) does not imply Strong CG. This is a result of Ishiu–P. Larson ([11]).

[4] We will deal with these posets in Section 3. There is also a (more) natural proper poset for adding, by initial segment, a club killing a potential $\mho$–sequence, but in order to kill a VWCG–sequence one seems to need finite approximations.

extension contains a club in $\mathbf{V}$. In particular, all these guessing principles P are preserved by c.c.c. forcing, and so they are consistent with $2^{\aleph_0}$ large. In particular, no forcing axiom MA_λ implies the negation of even Strong CG. Of course MA_{ω_1} implies neither VWCG nor $\mho$ since MA_{ω_1} follows from BPFA.

H. Sakai shows in [18] that the finite–support product $\mathsf{Add}(\omega,\,\theta)$ of θ copies of Cohen forcing always preserves $\neg\,$CG. A refined version of this argument shows that $\mathsf{Add}(\omega,\,\theta)$ preserves also $\neg\,$KA (cf. Lemma 1.14). On the other hand, it is not hard to see that Cohen forcing adds both a WCG–sequence and a $\mho$–sequence (cf. [13]). A consequence of these two facts together is that one can always force $2^{\aleph_0}$ large + $\mathrm{FA}(\mathsf{Add}(\omega,\,\lambda))_\mu$ for all $\lambda,\,\mu<2^{\aleph_0}$ + $\neg\,$CG + WCG + $\mathfrak{b}=\omega_1$ + KA. For this, start with a model of KA + $\neg\,$CG (cf. [10]) and add many Cohen reals to it. And of course one can also force $2^{\aleph_0}$ large + $\mathrm{FA}(\mathsf{Add}(\omega,\,\lambda))_\mu$ for all $\lambda,\,\mu<2^{\aleph_0}$ + $\neg\,$KA + WCG + $\mho$ + $\mathfrak{b}=\omega_1$: Start with a model of $\neg\,$KA and add many Cohen reals.

A forcing notion $\mathcal{P}$ is ${}^\omega\omega$–bounding if any $f:\omega\longrightarrow\omega$ in any extension by $\mathcal{P}$ is dominated by some $g:\omega\longrightarrow\omega$ in the ground model. It is not difficult to see that any c.c.c. ${}^\omega\omega$–bounding forcing preserves $\neg$WCG and $\neg\mathrm{VWCG}_\lambda$ for every λ:

Lemma 1.8. *(Hrušák) Given a limit ordinal $\alpha<\omega_1$ and $X\subseteq\alpha$, a set of ordinals of order type ω cofinal in α in an extension by an ${}^\omega\omega$–bounding forcing notion, X is covered by a set Y in $\mathbf{V}$ of order type ω and with $sup(Y)=\alpha$. Hence, If $\langle A_i\mid i<\lambda\rangle$ is a sequence of subsets of ω_1 of order type ω in an extension by a c.c.c. ${}^\omega\omega$–bounding forcing notion, then there is a sequence $\langle B_i\mid i<\lambda\rangle\in\mathbf{V}$ of subsets of ω_1 of order type ω such that $sup(B_i)=sup(A_i)$ and $|A_i\setminus B_i|<\aleph_0$ for every i.*

Proof. For the first assertion, let $(\alpha_n)_{n<\alpha}\in\mathbf{V}$ be a strictly increasing sequence converging to α and let $(h_n)_{n<\omega}\in\mathbf{V}$ be such that $h_n:\omega\longrightarrow[\alpha_n,\,\alpha_{n+1})$ is a surjection for every n. In the extension, let $f:\omega\longrightarrow\omega$ be given by $f(n)=\min\{m\mid X\cap[\alpha_n,\,\alpha_{n+1})\subseteq h_n``m\}$, and let $g:\omega\longrightarrow\omega$ be a function in $\mathbf{V}$ dominating f. It suffices to define Y such that $Y\cap[\alpha_n,\,\alpha_{n+1})=\{h_n(0),\ldots,h_n(g(n))\}$. Now we can prove the second assertion by a standard diagonalization argument using the c.c.c. □

It follows of course that we can always force $2^{\aleph_0}$ large + $\mathrm{FA}(\mathcal{B}_\lambda)_\mu$ for all $\lambda,\,\mu<2^{\aleph_0}$ + $\neg\,$VWCG, where $\mathcal{B}_\lambda$ denotes the measure algebra with λ–many generators: Start with a model of $\neg\,$VWCG and add lots of random reals to it.

At this point one would naturally ask whether MA_λ, for any $\lambda > \aleph_1$, implies any nontrivial weak club–guessing principle.[5] This question was answered negatively in [7]:

Definition 1.9. ([7]) A poset $\mathcal{P}$ has the $\aleph_{1.5}$*–c.c.* if for every regular cardinal $\lambda \geq |\mathsf{TC}(\mathcal{P})|^+$ there is a club $D \subseteq [H(\lambda)]^{\aleph_0}$ such that for every finite $\{N_i \mid i < n\} \subseteq D$ and every $p \in \mathcal{P}$, if $p \in N_j$ for some j such that $\delta_{N_j} = \min\{\delta_{N_i} \mid i < n\}$, then there is some condition extending p and $(N_i, \mathcal{P})$–generic for all i.

Also, given any cardinal λ, $\mathrm{MA}^{1.5}_\lambda$ is $\mathrm{FA}(\{\mathcal{P} \mid \mathcal{P}$ has the $\aleph_{1.5}$–c.c.$\})_\lambda$, and $\mathrm{MA}^{1.5}$ is $\mathrm{MA}^{1.5}_\lambda$ for all $\lambda < 2^{\aleph_0}$.

Every c.c.c. poset has the $\aleph_{1.5}$–c.c. and every $\aleph_{1.5}$–c.c. poset has the $\aleph_2$–c.c. and is proper. Also, for every cardinal λ, $\mathrm{MA}^{1.5}_\lambda$ implies $(\cdot)^\tau_\lambda$ for all $\tau < \omega_1$. These facts are proved in [7].

Theorem 1.10. *([7]) (CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa \mid \mathrm{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–c.c. such that the following statements hold in the generic extension by $\mathcal{P}$.*

(1) $2^{\aleph_0} = \kappa$
(2) $\mathrm{MA}^{1.5}$

The following strengthening of having the $\aleph_{1.5}$–c.c. is defined in [6].

Definition 1.11. A poset $\mathcal{P}$ is *finitely proper* iff for every regular regular cardinal $\lambda \geq |\mathsf{TC}(\mathcal{P})|^+$ there is a club $D \subseteq [H(\lambda)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq D$ and every $p \in \mathcal{P} \cap \bigcap \mathcal{N}$ there is a condition extending p which is $(N, \mathcal{P})$–generic for every $N \in \mathcal{N}$.

Thus, the different between having the $\aleph_{1.5}$–c.c. and being finitely proper is that, given a finite subset $\mathcal{N}$ of a suitable club of structures, in the former case we are requiring existence of extensions q of p which are generic for all members of $\mathcal{N}$ only for p's belonging to some $N \in \mathcal{N}$ of minimal height, whereas in the latter case p has to be in all members of $\mathcal{N}$.

One prominent finitely proper poset is Baumgartner's forcing for adding a club of ω_1 with finite conditions ([8]). We will denote this poset by $\mathbb{B}$. The observations on $\mathbb{B}$ that follow are borrowed from [4]. Conditions in $\mathbb{B}$ are finite functions $p \subseteq \omega_1 \times \omega_1$ that can be extended to a normal function $F : \omega_1 \longrightarrow \omega_1$, and given two conditions p_0, p_1, p_1 is stronger than p_0 iff $p_0 \subseteq p_1$. Here, a function $F : \omega_1 \longrightarrow \omega_1$ is

[5]Note that one cannot hope to use a finite support iteration of c.c.c. forcings to answer this question negatively since every such iteration will force WCG if it has length at least ω (see [7]).

normal if it is strictly increasing and continuous. There is of course a natural correspondence between clubs of ω_1 and normal functions $F : \omega_1 \longrightarrow \omega_1$ which sends a club to its enumerating function and a normal function to its range. $\mathbb{B}$ is a finitely proper poset of size $\aleph_1$. On the other hand, it is nowhere c.c.c. (i.e., it is not c.c.c. below any condition) and, as Zapletal proved in [25], under PFA it is a minimal nowhere c.c.c. poset, in the sense that every nowhere c.c.c. poset adds a generic for $\mathbb{B}$. Also, if $P = \{p_\alpha \mid \alpha < \omega_1\}$ is a proper nowhere c.c.c. forcing notion adding a club $C \subseteq \omega_1$ such that for all $\alpha \in C$, $\dot{G} \cap \{p_\beta \mid \beta < \alpha\}$ is generic for $\{p_\beta \mid \beta < \alpha\}$ (where $\dot{G}$ denotes the generic filter), then $\mathrm{RO}(P) = \mathrm{RO}(\mathbb{B})$ ([24]). Let $\mathrm{FA}(\mathbb{B})$ denote $\mathrm{FA}(\mathbb{B})_\lambda$ for all $\lambda < 2^{\aleph_1}$.

Given a partial order $\mathcal{P}$, let $\mathfrak{m}(\mathcal{P})$ be the minimal size of a family $\mathcal{D}$ of dense subsets of $\mathcal{P}$ such that there is no filter $G \subseteq \mathcal{P}$ intersecting all members of $\mathcal{D}$. There is a natural topology on the club filter $\mathcal{C}_{\omega_1}$ on ω_1 such that $\mathbf{cov}(\mathcal{C}_{\omega_1}) = \mathfrak{m}(\mathbb{B})$ if $\mathfrak{m}(\mathbb{B}) > \aleph_1$ (where $\mathcal{C}_{\omega_1}$ is endowed with this topology and where for a topological space X, $\mathbf{cov}(X)$, the covering number of X, is the minimal size of a collection of nowhere dense subsets of X whose union is X). This is of course the topology whose basis is given by the conditions in $\mathbb{B}$; in other words, a basis for this topology is given by $\{B_p \mid p \in \mathbb{B}\}$, where B_p is the set of all clubs $C \subseteq \omega_1$ such that $p \subseteq F$, where F is the enumerating function of C. The proof of this is a standard translation exercise between topological notions and order–theoretical notions, and is identical to the proof that $\mathfrak{m}(\text{Cohen})$ is the covering number of the meagre ideal for the Baire space. The assumption $\mathfrak{m}(\mathbb{B}) > \aleph_1$ is needed only for the proof of $\mathfrak{m}(\mathbb{B}) \leq \mathbf{cov}(\mathcal{C}_{\omega_1})$: Given $\kappa < \mathfrak{m}(\mathbb{B})$ and a collection $\{X_i \mid i < \kappa\}$ of closed nowhere dense subsets of $\mathcal{C}_{\omega_1}$, $D_i = \{p \in \mathbb{B} \mid B_p \cap X_i = \emptyset\}$ is a dense open subset of $\mathbb{B}$. Let $\{E_\nu \mid \nu < \omega_1\}$ be a set of dense subsets of $\mathbb{B}$ such that $\bigcup \mathrm{range}(G)$ is a club of ω_1 for every filter $G \subseteq \mathbb{B}$ meeting all E_ν. Since $\mathfrak{m}(\mathbb{B}) > \aleph_1$, we can find a filter $G \subseteq \mathbb{B}$ meeting all E_ν and all D_i. It follows then that $C = \bigcup \mathrm{range}(G)$ is a club of ω_1 such that $C \notin X_i$ of all i.

Of course there is nothing special about Cohen forcing or Baumgartner's forcing in any of these translations; in fact, a similar characterisation can be always obtained for $\mathfrak{m}(\mathcal{P})$ for any poset $\mathcal{P}$. What is nice about Cohen forcing and Baumgartner's forcing (in the case $\mathfrak{m}(\mathbb{B}) > \aleph_1$) is the appealing appearance of the topological side of the translation.

The assumption $\mathfrak{m}(\mathbb{B}) > \aleph_1$ in the above characterisation is necessary:

Fact 1.12. KA *implies* $\boldsymbol{cov}(\mathcal{C}_{\omega_1}) = \aleph_0$.

Proof. Let $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ be a KA–sequence. For every $n < \omega$, let D_n be the set of $\mathbb{B}$–conditions p such that for every $\delta \in \mathrm{dom}(p)$, if δ is perfect and $p(\delta) = \delta$, then there is some $m \geq n$ and some $\xi \in \mathrm{dom}(p) \cap \delta$ such that $p(\xi) < C_\delta(m)$ and $p(\xi + 1) > C_\delta(m + 1)$. D_n is clearly a dense subset of $\mathbb{B}$, and therefore the set X_n of $C \in \mathcal{C}_{\omega_1}$ such that $p \nsubseteq \tilde{C}$ for any $p \in D_n$, where $\tilde{C}$ is the enumerating function of C, is nowhere dense in $\mathcal{C}_{\omega_1}$. Now, if $C \subseteq \omega_1$ is a club, then there is some $n < \omega$ such that $C \in X_k$ for every $k \geq n$ since $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ is a KA–sequence. □

Thus, $\mathbf{cov}(\mathcal{C}_{\omega_1}) = \aleph_0$ can be regarded as a (very) weak club–guessing principle that might be worth investigating.

As we have seen, all forcing axioms of the form MA_λ are compatible with even the strongest club–guessing principle we are considering in this paper (namely Strong CG). This is not true for $\mathrm{FA}(\mathbb{B})_\lambda$: Let us say that a set $\mathcal{C}$ of subsets of ω_1 of order type ω is a KA set if for every club $D \subseteq \omega_1$ there is some $C \in \mathcal{C}$ such that $D \cap [C(n), C(n+1)) \neq \emptyset$ for a tail of $n < \omega$. Given a cardinal λ, let KA_λ be the assertion that there is a KA set of size at most λ. It is not difficult to see that $\mathbb{B}$ destroys every KA–sequence from the ground model. In particular we obtain the following.

Proposition 1.13. *For every cardinal* λ, $\mathrm{FA}(\mathbb{B})_\lambda$ *implies* $\neg\,\mathrm{KA}_\lambda$.

Given a cardinal λ, let CG_λ denote the weakening of CG saying that there is a set $\mathcal{C}$ of size at most λ consisting of subsets of ω_1 of order type ω and such that for every club $D \subseteq \omega_1$ there is some $C \in \mathcal{C}$ with $C \subseteq D$. Of course CG_λ implies KA_λ.

By essentially the same argument as in [18] one can show that $\mathsf{Add}(\omega, \theta)$ preserves $\neg\,\mathrm{CG}_\lambda$. Also, by refining the argument from [18], one can establish the following preservation result ([4]), the proof of which we include here for completeness:

Lemma 1.14. $\mathsf{Add}(\omega, \theta)$ *preserves* $\neg\,\mathrm{KA}_\lambda$.

Proof. Let X be any set and let $P(X)$ be the set of finite functions $p \subseteq (X \times \omega) \times 2$ ordered by reverse inclusion. Let $\langle \dot{A}_i \mid i < \lambda\rangle$ be a sequence of $P(X)$–names for subsets of ω_1 of order type ω. It suffices to show that if this sequence witnesses KA_λ in the generic extension given by $P(X)$, then KA_λ also holds in the ground model. The first observation is that for every i there is a countable $Y_i \subseteq X$ such that $\dot{A}_i$ is in fact a $P(Y_i)$–name and such that for every $\alpha < \omega_1$ there is some $p \in P(X)$ such that $p \Vdash_{P(X)} \check{\alpha} \in \dot{A}_i$ if and only if there is some $p \in P(Y)$ such that $p \Vdash_{P(Y)} \check{\alpha} \in \dot{A}_i$.

For every i let $(p^i_n)_{n<\omega}$ be an enumeration of $P(Y_i)$. Also, for every $n < \omega$, if there is some $\sigma \in \omega_1$ such that $p^i_n \Vdash_{P(Y_i)} \sup(\dot{A}_i) = \sigma$, then let $\mathcal{X}^i_n$ be a set of pairwise compatible $P(Y_i)$–conditions extending p^i_n and such that $\{\xi < \sigma \mid p \Vdash_{P(Y_i)} \xi \in \dot{A}_i \text{ for some } p \in \mathcal{X}^i_n\}$ is cofinal in σ.[6]

Given any club $C \subseteq \omega_1$ there are $i < \lambda$, $\sigma < \omega_1$, $\gamma < \sigma$ and $n^C < \omega$ such that $p^i_{n^C}$ forces in $P(Y_i)$ that $\sup(\dot{A}_i) = \sigma$ and that $[\alpha, \alpha') \cap C \neq \emptyset$ for every two consecutive points α, α' of $\dot{A}_i$ above γ. Let $n^* < \omega$ be such that the set $\mathcal{C}$ of clubs C such that $n^C = n^*$ is $\subseteq$–dense in the set of all clubs of ω_1. For every $i < \lambda$, if there is a σ such that $p^i_{n^*} \Vdash_{P(Y_i)} \sup(\dot{A}_i) = \sigma$, then let $B_i = \{\xi < \sigma \mid p \Vdash_{P(Y_i)} \xi \in \dot{A}_i \text{ for some } p \in \mathcal{X}^i_{n^*}\}$.

By $\subseteq$–density of $\mathcal{C}$ it suffices to show that if $C \in \mathcal{C}$, then there is some $i < \lambda$ such that $p^i_{n^*} \Vdash \sup(\dot{A}_i) = \sigma$ for some σ, and such that $[\beta, \beta') \cap C \neq \emptyset$ for every two consecutive points $\beta < \beta'$ in B_i above some $\gamma < \sigma$. But this is true by the previous paragraph and the definition of B_i since all conditions in $\mathcal{X}^i_{n^*}$ extend $p^i_{n^*}$ and are pairwise compatible. □

It follows that if, say, GCH holds and $\omega_2 \leq \kappa \leq \lambda$ are successor cardinals, then there is a proper $\aleph_2$–c.c. poset producing a model of $2^{\aleph_0} = \lambda$ + $\mathrm{FA}(\mathsf{Add}(\omega, \mu))_{\lambda'}$ for all λ', $\mu < \lambda$ + $\neg\,\mathrm{KA}_{\kappa'}$ for any $\kappa' < \kappa$ +WCG +$\mho$+ $\mathfrak{b} = \omega_1$ (this result will be partially improved in Theorem 1.16). For this, first force $2^{\aleph_0} = \kappa + \mathrm{MA}^{1.5}$ by a proper $\aleph_2$–c.c. poset as in Theorem 1.10 and then force with $\mathsf{Add}(\omega, \lambda)$. The first forcing kills $\mathrm{KA}_{\kappa'}$ for all $\kappa' < \kappa$, and the second forcing preserves $\neg\,\mathrm{KA}_{\kappa'}$ for all $\kappa' < \kappa$ and forces $2^{\aleph_0} = \lambda$ + WCG +$\mathfrak{b} = \omega_1$.

In this paper we focus on the construction of models separating club–guessing principles in which $2^{\aleph_0}$ is arbitrarily large and in which some relatively large fragment of $\mathrm{MA}^{1.5}$ (say, comprising $\mathrm{MA} + \mathrm{FA}(\mathbb{B})$) holds. Our main theorems are the following.

Theorem 1.15. *(CH) Suppose there is a strong $\mho$–sequence $\mathcal{G}$. Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–c.c. such that the following statements hold in $V^{\mathcal{P}}$.*

(1) *$\mathcal{G}$ is a $\mho$–sequence.*
(2) *$(\cdot)^\tau_\lambda$ for all $\tau < \omega_1$ and $\lambda < 2^{\aleph_0}$.*

[6]The introduction of the $\mathcal{X}^i_n$'s is the new ingredient with respect to the proof in [18].

(3) MA
(4) FA($\mathbb{B}$)
(5) $2^{\aleph_0} = \kappa$

Theorem 1.16. *(*CH*) Suppose there is a strong* WCG*–sequence $\mathcal{A}$. Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–c.c. such that the following statements hold in $V^{\mathcal{P}}$.*

(1) $\mathcal{A}$ *is a* WCG*–sequence.*
(2) $\neg\mho$
(3) MA
(4) FA($\mathbb{B}$)
(5) $2^{\aleph_0} = \kappa$

By Fact 1.7 and the paragraph before Fact 1.7, the hypotheses of Theorems 1.15 and 1.16 can always be forced.

Both theorems above can be proved starting from the weaker hypothesis that $\mathcal{G}$ (resp., $\mathcal{A}$) is defined only on a stationary subset of ω_1 (for Theorem 1.15 and Theorem 1.16, respectively). Here, the relevant object $\mathcal{R}$ being strongly guessing in the appropriate sense means of course that for every club C the guessing occurs for all ordinals in $D \cap \mathrm{dom}(\mathcal{R})$ for some club $D \subseteq \omega_1$.

Finally, we should point out that other separations of consequences of $\mathrm{MA}^{1.5}$, in the presence of $2^{\aleph_0}$ large, have been obtained also by other people using iterations with symmetric systems of structures as side conditions and considering instances from a restricted class of $\aleph_{1.5}$–c.c. forcings only (i.e., as in the proof of Theorem 1.15). Specifically, Yorioka ([23]) builds a model of $\neg\mho + \neg\,\mathrm{WCG} + \mathfrak{p} = \mathsf{add}(\mathcal{N}) = 2^{\aleph_0} +$ $2^{\aleph_0}$ large in which there is a destructible (ω_1, ω_1^*)–gap in $({}^{\omega}\omega, <^*)$.

2. Forcing with symmetric systems

For this section, let κ be a cardinal such that $\mathrm{cf}(\kappa) \geq \omega_2$.

Definition 2.1. ([6]; cf. [21], [14], [1]) Let $P \subseteq H(\kappa)$, and let $\mathcal{N}$ be a collection of countable subsets of $H(\kappa)$. We will say that $\mathcal{N}$ *is a P–symmetric system* iff the following holds.

(A) For every $N \in \mathcal{N}$, $(N, \in, P)$ is an elementary substructure of $(H(\kappa), \in, P)$.
(B) Given N, $N' \in \mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a (unique) isomorphism
$$\Psi_{N,N'} : (N, \in, P) \longrightarrow (N', \in, P)$$
Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(C) For all M, N in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

(D) For all M, N and N' in $\mathcal{N}$, if $M \in N$ and $\delta_{N'} = \delta_N$, then $\Psi_{N,N'}(M) \in \mathcal{N}$.

In (A) in the above definition, and elsewhere, we will tend to refer to structures $(N, \in, P \cap N)$ by the simpler expression $(N, \in, P)$. Also, given any two structures N, N', if there is a (unique) isomorphism between $(N, \in)$ and $(N', \in)$, then we denote this isomorphism by $\Psi_{N,N'}$.

Given $P \subseteq H(\kappa)$, there is a natural forcing notion $\mathcal{S}_P$ for adding, by initial approximations, a symmetric P–system $\mathcal{N}$ such that $\bigcup \mathcal{N} = H(\kappa)$: $\mathcal{S}_P$ is just the set of finite P–symmetric systems, ordered by reverse inclusion.

The following lemma is proved in [6] (Lemma 2.3).

Lemma 2.2. *Let $P \subseteq H(\kappa)$, let $\mathcal{N}$ be a P–symmetric system, and let $N \in \mathcal{N}$. Then the following holds.*

(i) *$\mathcal{N} \cap N$ is a P–symmetric system.*

(ii) *If $\mathcal{W} \subseteq N$ is a P–symmetric system and $\mathcal{N} \cap N \subseteq \mathcal{W}$, then*

$$\mathcal{V} := \mathcal{N} \cup \{\Psi_{N,N'}(W) \mid W \in \mathcal{W},\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

is a P–symmetric system.

Recall the following strengthening of the notion of properness: A poset $\mathbb{P}$ is *strongly proper* iff for every large enough regular θ there are club–many countable $N^* \preccurlyeq H(\theta)$ such that every $p \in N^* \cap \mathbb{P}$ can be extended to a strongly $(N^*, \mathbb{P})$–generic condition. Here, a condition q is said to be *strongly $(N^*, \mathbb{P})$–generic* if for every dense subset D of $\mathbb{P} \cap N$, D is predense in $\mathbb{P}$ below q.

We now have the following corollary from Lemma 2.2.

Corollary 2.3. *$\mathcal{S}_P$ is strongly proper. In fact, suppose that θ is a regular cardinal and N^* is a countable elementary substructure of $H(\theta)$ such that $P \in N^*$. Then, letting $N = N^* \cap H(\kappa)$, the following conditions hold for every $\mathcal{N} \in \mathcal{S}_P$.*

(1) *If $\mathcal{N} \in N$, then $\mathcal{N} \cup \{N\} \leq_{\mathcal{S}_P} \mathcal{N}$.*

(2) *If $N \in \mathcal{N}$, then $\mathcal{N}$ is strongly $(N^*, \mathcal{S}_P)$–generic.*

Proof. (1) is obvious. For (2), let E be a dense subset of $\mathcal{S}_P \cap N$ and let $\mathcal{N}'$ be any $\mathcal{S}_P$–condition extending $\mathcal{N}$. It suffices to show that there is some condition in E compatible with $\mathcal{N}'$ in $\mathcal{S}_P$. Notice that $\mathcal{N}' \cap N \in \mathcal{S}_P \cap N$ by Lemma 2.2 (i). Hence, we may find a condition $\mathcal{N}^\circ \in E \cap N$ extending $\mathcal{N}' \cap N$. Now let

$$\mathcal{N}^* = \mathcal{N}' \cup \{\Psi_{N,\overline{N}}(M) \mid M \in \mathcal{N}^\circ,\, \overline{N} \in \mathcal{N},\, \delta_{\overline{N}} = \delta_N\}$$

By Lemma 2.2 (ii) we have that $\mathcal{N}^*$ is a condition in $\mathcal{S}_P$ extending both $\mathcal{N}'$ and $\mathcal{N}^\circ$. $\square$

The following is essentially Lemma 2.4 in [6].

Lemma 2.4. *Let $P \subseteq H(\kappa)$ and let $\mathcal{N}_0 = \{N_i^0 \mid i < \mu\}$ and $\mathcal{N}_1 = \{N_i^1 \mid i < \mu\}$ be P–symmetric systems. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = X$ and that there is an isomorphism Ψ between the structures*

$$\langle \bigcup_{i<\mu} N_i^0, \in, P, X, N_i^0 \rangle_{i<\mu}$$

and

$$\langle \bigcup_{i<\mu} N_i^1, \in, P, X, N_i^1 \rangle_{i<\mu}$$

which is the identity on X. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a P–symmetric system.

Recall that, given a cardinal λ, a poset $\mathbb{P}$ is λ–Knaster if for every $X \in [\mathbb{P}]^\lambda$ there is some $Y \in [X]^\lambda$ consisting of pairwise compatible conditions. The following lemma is easy to prove using Lemma 2.4 together with standard Δ–system arguments (cf. [6], Lemma 3.9).

Lemma 2.5. *(CH) Let $P \subseteq H(\kappa)$. If there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, then $\mathcal{S}_P$ is $\aleph_2$–Knaster.*

Note that if $\mathbb{P}$ is a non-atomic strongly proper forcing, then $\mathbb{P}$ adds new reals and all of them are in fact Cohen reals. This is simply because if $\dot{r}$ is a $\mathbb{P}$–name and p forces that $\dot{r}$ is a real, then we can find a countable elementary substructure N^* containing all the relevant objects and a strongly $(N^*, \mathbb{P})$–generic condition q extending p. Then q forces that $\dot{r}$ is decided by $N^* \cap \dot{G}$, and that $N^* \cap \dot{G}$ is generic over the countable non-atomic poset $N^* \cap \mathbb{P}$, which is forcing–equivalent to Cohen forcing. It follows from this, together with Corollary 2.3, that $\mathcal{S}_P$ adds Cohen reals. It is also not difficult to see that forcing with $\mathcal{S}_P$ adds in fact at least $\aleph_1$–many Cohen reals.

Fact 2.6. *For every $P \subseteq H(\kappa)$, $\mathcal{S}_P$ adds $\aleph_1$–many Cohen reals.*

Proof. Let G be $\mathcal{S}_P$–generic. Given any $\delta < \omega_1$, if there is some containing $N^* \preccurlyeq H(|H(\kappa)|^+)$ containing P and such that, letting $N = N^* \cap H(\kappa)$, $\delta_N = \delta$ and $N \in \bigcup G$, then let $r_\delta \in \mathbf{V}[G \cap (\mathcal{S}_P \cap N)]$ be a Cohen real over $\mathbf{V}$ such that $\mathcal{S}_P \cap N \in \mathbf{V}[r_\delta]$. Such a Cohen real can be found since $\mathcal{S}_P \cap N$ is clearly non-atomic.

To see that $r_{\delta_0} \neq r_{\delta_1}$ if $\delta_0 < \delta_1$ are as above, note that that if $N_0 \in N_1$ are structures as above such that $\delta_{N_0} = \delta_0$ and $\delta_{N_1} = \delta_1$, then $\mathcal{S}_P \cap N_1$ is forcing–equivalent to $(\mathcal{S}_P \cap N_0) * \dot{\mathcal{Q}}$, where $\dot{\mathcal{Q}}$ is a $\mathcal{S}_P \cap N_0$–name for the suborder of $\check{\mathcal{S}}_P$ consisting of those $\mathcal{N} \in \check{\mathcal{S}}_P$ such that $\mathcal{N} \cap N_0 \in \dot{G}_0$ (where $\dot{G}_0$ of course denotes a name for the $\mathcal{S}_P \cap N_0$–generic filter). But now we are done since $\dot{\mathcal{Q}}$ is forced to be non-atomic. $\square$

What is perhaps more surprising is that if CH holds, then $\mathcal{S}_P$ does not add more than $\aleph_1$–many reals.

Proposition 2.7. *(*CH*) Let $P \subseteq H(\kappa)$. If there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, then $\mathcal{S}_P$ preserves* CH.

Proof. Suppose $\dot{r}_\alpha$ (for $\alpha < \omega_2$) are $\mathcal{S}_P$–names for members of ${}^\omega 2$ and $\mathcal{N} \in \mathcal{S}_P$ forces $\dot{r}_\alpha \neq \dot{r}_{\alpha'}$ for all $\alpha < \alpha' < \omega_2$. By the $\aleph_2$–c.c. of $\mathcal{S}_P$ we may assume that each $\dot{r}_\alpha$ is in $H(\kappa)$. Let θ be a regular cardinal such that $\mathcal{S}_P \in H(\theta)$. For each α let N_α be such that $\{\mathcal{N}, \dot{r}_\alpha\} \in N_\alpha$ and N_α is a countable elementary substructure of $(H(\kappa), \in, P, \mathcal{S}_P)$. We may also assume that for each α there is a countable $N^*_\alpha \prec H(\theta)$ such that $N_\alpha = H(\kappa) \cap N^*_\alpha$. By CH we may find distinct α, α' such that $(N_\alpha, \in, P, \mathcal{S}_P, \mathcal{N}, \dot{r}_\alpha)$ and $(N_\alpha, \in, P, \mathcal{S}_P, \mathcal{N}, \dot{r}_\alpha)$ are isomorphic, and we may also assume that the unique isomorphism Ψ between these structures fixes $N_\alpha \cap N_{\alpha'}$. By Lemma 2.4 we know that $\mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$ is an $\mathcal{S}_P$–condition, and by Corollary 2.3, $\mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$ is $(N^*_\alpha, \mathcal{S}_P)$–generic and $(N^*_{\alpha'}, \mathcal{S}_P)$–generic. Note that for every $n < \omega$ and every condition $\mathcal{N}'$ $\mathcal{S}_P$–extending $\mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$ there are conditions $\mathcal{N}''$, $\mathcal{M}$ and $\epsilon \in 2$ such that $\mathcal{M} \in N_\alpha$, $\mathcal{M} \Vdash_{\mathcal{S}_P} \dot{r}_\alpha(n) = \epsilon$, and $\mathcal{N}''$ is a common $\mathcal{S}_P$–extension of $\mathcal{M}$ and $\mathcal{N}'$ (this is true since $\mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$ is $(\mathcal{S}_P, N^*_\alpha)$–generic). By clause (D) in Definition 2.1, $\mathcal{N}''$ also $\mathcal{S}_P$–extends $\Psi(\mathcal{M}) \in N_{\alpha'}$. By correctness of Ψ with respect to the predicate $\mathcal{S}_P$, $\Psi(\mathcal{M}) \Vdash_{\mathcal{S}_P} \Psi(\dot{r}_\alpha)(n) = \dot{r}_{\alpha'}(n) = \epsilon$. Since $\mathcal{N}'$ and n were arbitrary this shows that $\mathcal{N} \cup \{N_\alpha, N_{\alpha'}\}$ forces that $\dot{r}_\alpha = \dot{r}_{\alpha'}$, which is a contradiction. $\square$

3. Proving Theorems 1.15 and 1.16

The following notation will be used throughout the paper: If q is an ordered pair, we denote the first and the second components of q by F_q and Δ_q, respectively. Also, if q is an ordered pair such that F_q is a function and Δ_q is a binary relation with range$(\Delta_q) \subseteq$ Ord and ξ is an ordinal, *the restriction of q to ξ*, denoted by $q|_\xi$, is defined as the pair

$$q|_\xi := (F_q \restriction \xi,\ \{(N, \min\{\beta,\, \xi\}) \mid (N, \beta) \in \Delta_q\})$$

Similarly, if q is an ordered pair such that F_q is a function and Δ_q is a set of triples $(M, \vec{\mathcal{W}}, \beta)$ with $\beta \in$ Ord and ξ is an ordinal, *the restriction of q to ξ*, again denoted by $q|_\xi$, is defined as the pair

$$q|_\xi := (F_q \restriction \xi,\ \{(N, \vec{\mathcal{W}}, \min\{\beta,\, \xi\}) \mid (N, \vec{\mathcal{W}}, \beta) \in \Delta_q\})$$

We will be using instances of the following forcing for adding, by initial approximations, a club of ω_1 avoiding a given collection of subsets of ω_1 in the sense of $(\cdot)^\tau_\lambda$ (cf. [7]).

Definition 3.1. Let $\tau < \omega_1$ and let $\mathcal{A} = \langle A_i \mid, i < \lambda\rangle \subseteq \mathcal{P}(\omega_1)$ be such that each A_i is τ–thin. Let $\mathbb{B}^\tau_{\mathcal{A}}$ consist of all pairs (f, b) such that

(1) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function such that $\text{rank}(f(\nu), f(\nu)) \geq \max\{\tau, \nu\}$ for every $\nu \in \text{dom}(f)$,
(2) b is a finite function with $\text{dom}(b) \subseteq \lambda$ and $b(i) \in [\text{range}(f)]^{<\omega}$ for every $i \in \text{dom}(b)$, and
(3) for every $i \in \text{dom}(b)$, $\text{range}(f) \cap A_i = b(i)$.

Given $\mathbb{B}^\tau_{\mathcal{A}}$–conditions (f_0, b_0) and (f_1, b_1), (f_1, b_1) extends (f_0, b_0) if $f_0 \subseteq f_1$ and $b_0 \subseteq b_1$.

Remark 3.2. If $\mathcal{N}$ is a finite set of countable elementary substructure of $H(\theta)$, for a large enough θ, $\mathbb{B}^\tau_{\mathcal{A}} \in N$, and $(f, b) \in \bigcap \mathcal{N} \cap \mathbb{B}^\tau_{\mathcal{A}}$, then $(f \cup \{\langle \delta_N, \delta_N\rangle : N \in \mathcal{N}\}, b)$ is $(N, \mathbb{B}^\tau_{\mathcal{A}})$–generic for each $N \in \mathcal{N}$.

Proof. We prove the remark only for the case when $\mathcal{N}$ contains only one member N (the proof of the general case is essentially the same).

Suppose that (f_1, b_1) extends $(f \cup \{\langle \delta_N, \delta_N\rangle\}, b)$ and let E be an open dense subset of $\mathbb{B}^\tau_{\mathcal{A}}$ in N. We must show that (f_1, b_1) is compatible with a condition in $E \cap N$.

Without loss of generality we may assume that $(f_1, b_1) \in E$. Let $\mu = \max(\text{range}(f_1 \restriction \delta_N))$, let $\sigma = f_1^{-1}(\mu) + 1$, and let $g : \omega_1 \setminus (\mu + 1) \longrightarrow \omega_1$ be the function sending each $\nu \in \omega_1 \setminus (\mu + 1)$ to the least ξ for which there is a condition $(f', b') \in E$ such that

(a) (f', b') extends $(f_1 \restriction N, b_1 \restriction N)$,
(b) $f' \restriction \sigma = f \restriction \sigma$,
(c) $\xi > \nu$, and
(d) ξ is the least ordinal in $\text{range}(f')$ strictly above μ.

Note that for every $\nu \in \delta_N \setminus (\mu + 1)$, δ_N and (f_1, b_1) together witness that the set of pairs $(\xi, (f', b'))$ satisfying (a)–(d) for ν is nonempty. Hence g is a well–defined function. Note also that g, being definable from the condition $(f_1 \restriction N, b_1 \restriction N)$, is in N since $f_1(\delta_N) = \delta_N$ and therefore $(f_1 \restriction N, b_1 \restriction N) \in N$. It follows that the club C of $\eta < \omega_1$ such that $g``\eta \subseteq \eta$ is also in N. Now, C has order type ω_1, and therefore $C \cap \delta_N$ has order type δ_N by correctness of N. Hence, by the finiteness of the domain of b_1 together with finitely many applications of Fact 1.3, we may find some $\eta \in \delta_N \cap C$ and some $\nu < \eta$ such that $[\nu, \eta] \cap A_\delta = \emptyset$ for every $\delta \in \text{dom}(b_1)$ such that $\delta \notin N$. But then, by definition of g together with the correctness of N there is some (f', b') in $N \cap E$ extending $(f_1 \restriction N, b_1 \restriction N)$ such that $f' \restriction \sigma = f_1 \restriction \sigma =$

$f_1 \restriction N$, range$(f') \subseteq \eta$, and such that the least ordinal in the range of f' strictly above μ is also above ν. But then $f_1 \cup f'$ is such that range$(f_1 \cup f') \cap A_\delta =$ range$(f_1) \cap A_\delta = b_1(\delta)$ for every $\delta \in \mathrm{dom}(b_1)$. It then follows that $(f_1 \cup f', b_1 \cup b')$ is a condition in $\mathbb{B}^\tau_{\mathcal{A}}$ extending both (f_1, b_1) and (f', b'). $\square$

Since $2^{<\kappa} = \kappa$, we can fix a function $\Phi : \kappa \longrightarrow H(\kappa)$ such that $\Phi^{-1}(x)$ is unbounded in κ for every $x \in H(\kappa)$. Let $\langle \theta_\alpha \mid \alpha \leq \kappa \rangle$ be the strictly increasing sequence of regular cardinals defined as $\theta_0 = |2^\kappa|^+$ and $\theta_\alpha = |2^{\sup\{\theta_\beta | \beta < \alpha\}}|^+$ if $\alpha > 0$. For each $\alpha \leq \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ and $\langle \theta_\beta \mid \beta < \alpha \rangle$. Let also $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \mid N^* \in \mathcal{M}^*_\alpha\}$ and note that if $\alpha < \beta$, then $\mathcal{M}^*_\alpha \in N^*$ for all $N^* \in \mathcal{M}^*_\beta$ such that $\alpha \in N^*$.

The forcing $\mathcal{P}$ witnessing Theorem 1.15 will be $(\mathcal{P}_\kappa, \leq_\kappa)$, where the sequence $\langle (\mathcal{P}_\alpha, \leq_\alpha) \mid \alpha \leq \kappa \rangle$ is defined as follows. To start with, $\mathcal{P}_0$ consists of all pairs of the form

(a) $(\emptyset, \{(N_i, 0) \mid i < m\})$, where $\{N_i \mid i < m\}$ is a finite Φ–symmetric system.

Given $\mathcal{P}_0$–conditions q and p, $q \leq_0 p$ iff $\Delta_p \subseteq \Delta_q$.

Let $\alpha = \sigma + 1$. Suppose that $\Phi(\sigma)$ is a $\mathcal{P}_\sigma$–name for either a c.c.c. forcing or that there is some $\tau < \omega_1$ and some $\mathcal{P}_\sigma$–name $\dot{\mathcal{A}}$ for a subset of $\mathcal{P}(\omega_1)$ all of whose members are τ–thin and such that $\Phi(\sigma)$ is a $\mathcal{P}_\sigma$–name for $\mathbb{B}^\tau_{\dot{\mathcal{A}}}$. Then we let $\dot{\mathcal{Q}}_\sigma = \Phi(\sigma)$. If $\Phi(\sigma)$ is not as above, then we let $\dot{\mathcal{Q}}_\sigma$ be a $\mathcal{P}_\sigma$–name for (say) trivial forcing on $\{0\}$.

If $\alpha \leq \kappa$ (regardless of whether α is a successor or a nonzero limit ordinal), the definition of $\mathcal{P}_\alpha$ is as follows. Conditions in $\mathcal{P}_\alpha$ are pairs of the form

$$q = (F_q,\, \Delta_q)$$

with the following properties.

$(b\,0)$ F_q is a finite function with $\mathrm{dom}(F_q) \subseteq \alpha$.

$(b\,1)$ Δ_q is of the form $\{(N_i, \beta_i) \mid i < m\}$ where, for all $i < m$, $\beta_i \leq \alpha \cap \sup(N_i \cap \kappa)$.

$(b\,2)$ For all $\xi < \alpha$, $q|_\xi \in \mathcal{P}_\xi$.

$(b\,3)$ If $\xi \in \mathrm{dom}(F_q)$, then $q|_\xi$ forces (in $\mathcal{P}_\xi$) that $F_q(\xi) \in \dot{\mathcal{Q}}_\xi$.

$(b\,4)$ If $\xi \in \mathrm{dom}(F_q)$, $\dot{\mathcal{Q}}_\xi$ is a $\mathcal{P}_\xi$–name for a poset of the form $\mathbb{B}^\tau_{\mathcal{A}}$ for some $\tau < \omega_1$ and some $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ all of whose members are τ–thin, $(N, \beta) \in \Delta_q$, $\beta \geq \xi + 1$, and $N \in \mathcal{M}_{\xi+1}$, then δ_N is a fixed point of the first component of $F_q(\xi)$.

Given conditions

$$q^\epsilon = (F_\epsilon,\, \{(N^\epsilon_i, \beta^\epsilon_i) \mid i < m_\epsilon\})$$

(for $\epsilon \in \{0,1\}$) in $\mathcal{P}_\alpha$, we will say that $q^1 \leq_\alpha q^0$ if and only if the following holds.

(c 1) For all $\xi < \alpha$, $q^1|_\xi \leq_\xi q^0|_\xi$.
(c 2) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and, for all $\xi \in \mathrm{dom}(F_0)$,

$$q^1|_\xi \Vdash_\xi F_1(\xi) \leq_{\dot{\mathcal{Q}}_\xi} F_0(\xi)$$

.

(c 3) For all $i < m_0$ there is some $\widetilde{\beta}_i \geq \beta^0_i$ such that $(N^0_i, \widetilde{\beta}_i) \in \Delta_{q^1}$.

Note that if $\alpha < \beta \leq \kappa$, then $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$; in fact, every $\mathcal{P}_\beta$–condition $q = (F, \{(N_j, \beta_j) \mid j < m\})$ such that $\mathrm{dom}(q) \subseteq \alpha$ and $\beta_j \leq \alpha$ for all j is also a $\mathcal{P}_\alpha$–condition and is in fact its restriction to α.

Also note that if $\alpha \leq \kappa$ and $q \in \mathcal{P}_\alpha$, then $\mathrm{dom}(\Delta_q) \subseteq \mathrm{dom}(\Delta_{q|_0})$ and $q|_0 \in \mathcal{P}_0$.

Finally note that if α is a nonzero limit ordinal, then a pair $q = (F_q, \Delta_q)$ is a $\mathcal{P}_\alpha$–condition if and only if it satisfies (b 0)–(b 2).

For every $\mathcal{P}_\kappa$–condition q and every $\xi \in \mathrm{dom}(F_q)$, if $\dot{\mathcal{Q}}_\xi$ is a $\mathcal{P}_\xi$–name for a forcing of the form $\mathbb{B}^\tau_{\mathcal{A}}$, for some $\tau < \omega_1$ and $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ as before, we denote by $f_{q,\xi}$ and $b_{q,\xi}$ the first and second components of $F_q(\xi)$, respectively.

The following lemmas are proved in [6] essentially.

Lemma 3.3. *Let $\alpha < \beta \leq \kappa$, let $r \in \mathcal{P}_\beta$ and let $q \in \mathcal{P}_\alpha$ be such that $q \leq_\alpha r|_\alpha$. Then $(F_q \cup (F_r \restriction [\alpha, \beta), \Delta_q \cup \Delta_r)$ is a condition in $\mathcal{P}_\beta$ extending both of q and r. In particular, $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.*

Lemma 3.4. *(*CH*) For every ordinal $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ has the $\aleph_2$–c.c.*

Lemma 3.5. *$\mathcal{P}_\kappa$ forces $2^{\aleph_0} = \kappa$.*

Definition 3.6. Given $\leq \alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure $N \prec H(\kappa)$, we will say that q is $(N, \mathcal{P}_\alpha)$–pre-generic in case

- $\alpha < \kappa$ and $(N, \alpha) \in \Delta_q$, or else
- $\alpha = \kappa$ and the pair $(N, \sup(N \cap \kappa)) \in \Delta_q$.

Lemma 3.7. *Suppose $\alpha \leq \kappa$ and $N^* \in \mathcal{M}^*_\alpha$. Let $N = N^* \cap H(\kappa)$. Then the following conditions hold.*

$(1)_\alpha$ *For every $q \in N$ there is some $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$–pre-generic.*
$(2)_\alpha$ *If $\mathcal{P}_\alpha \in N^*$ and $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$–pre-generic, then q is $(N^*, \mathcal{P}_\alpha)$–generic.*

Corollary 3.8. *For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is proper.*

The proof of Lemma 3.4 uses the fact that if q and q' are $\mathcal{P}_\alpha$–conditions, $\sigma \in \mathrm{dom}(F_q) \cap \mathrm{dom}(F_{q'})$, and f is such that $F_q(\sigma) = (f, b)$ and $F_{q'}(\sigma) = (f, b')$, for some b and b', then $q|_{\sigma+1}$ and $a'|_{\sigma+1}$ are compatible in $\mathcal{P}_{\sigma+1}$ whenever $q|_\sigma$ and $q'|_\sigma$ are compatible in $\mathcal{P}_\sigma$. Also, Lemma 3.7 holds thanks to all $\dot{\mathcal{Q}}_\xi$ being forced to be finitely proper (see Remark 3.2). Incidentally, these two remarks together show that already a very mild alteration to the main construction in [6] suffices for showing that no forcing axiom MA_κ implies weak club–guessing principles on ω_1.

It is easy to see that forcing with any instance of $\mathbb{B}^\tau_{\mathcal{A}}$ as in Definition 3.1 adds a generic filter for $\mathbb{B}$. By standard book-keeping arguments together with Lemma 3.4 it follows then that $\mathcal{P}$ forces MA + FA($\mathbb{B}$) + $(\cdot)^\tau_\lambda$ for all $\tau < \omega_1$ and $\lambda < \kappa$.

In order to complete the proof of Theorem 1.15 it therefore suffices to prove the following lemma.

Lemma 3.9. *Let $\alpha < \kappa$ and suppose $\mathcal{G} = \langle g_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ is a strong $\mho$–sequence. Let $n < \omega$ and suppose that M^* and M satisfy the following:*

1. *$M^* \in \mathcal{M}^*_{\alpha+2}$ and $M = M^* \cap H(\kappa)$.*
2. *$\dot{C} \in M$ is a $\mathcal{P}_\alpha$–name for a club of ω_1.*
3. *$(M, \alpha) \in \Delta_q$ for some $q \in G$.*

Then q forces that there is some $\varepsilon \in \dot{C} \cap \delta_M$ such that $g_{\delta_M}(\varepsilon) = n$.

Proof. The proof is by induction on α. The case $\alpha = 0$ follows immediately from Fact 1.5.

For the case when $\alpha = \sigma + 1$, let $q' \in \mathcal{P}_\alpha$ be an extension of q. Without loss of generality we may assume that $\sigma \in F_{q'}$ and that there is some $\tau < \omega_1$ and some $\mathcal{P}_\sigma$–name $\dot{\mathcal{A}}$ for a subset of $\mathcal{P}(\omega_1)$ all of whose members are τ–thin and such that $\dot{\mathcal{Q}}_\sigma$ is a $\mathcal{P}_\sigma$–name for $\mathbb{B}^\tau_{\dot{\mathcal{A}}}$ (the proof in the other cases is easier). We have that δ_M is a fixed point of $f_{q',\sigma}$. Let G be a $\mathcal{P}_\sigma$–generic with $q'|_\sigma \in G$ and let $\dot{\mathcal{A}}_G = (A_i \mid i < \lambda)$. By Lemma 3.7, G is also generic over M^* and we may therefore assume $\tau \in M$. We may assume that $f_{q',\sigma} \restriction \delta_M \neq \emptyset$. Let $\delta = \max(\mathrm{dom}(f_{q',\sigma} \restriction \delta_M)$. Now we consider the following function F with domain the open interval $(f_{q',\sigma}(\delta), \omega_1)$. Suppose ϱ is in this interval. Assume that there exists a condition t in $\mathcal{P}_\alpha$ with the following properties.

(a) $t|_\sigma \in G$
(b) $(f_{q',\sigma} \restriction \delta_M) \cup \{\langle \delta + 1, \varrho\rangle\} \subseteq f_{t,\sigma}$
(c) $f_{t,\sigma} \restriction (\delta + 1) = f_{q',\sigma} \restriction \delta_M$

(d) There is some $\nu \in \mathrm{dom}(f_{t,\sigma})$, $\nu > \delta + 1$, such that $f_{t,\sigma}(\nu) = \delta_N$ for some $N \in \mathcal{M}_{\alpha+1}$ such that $\dot{C} \in N$ and $(N, \alpha) \in \Delta_t$ (so, by Lemma 3.7, t forces $f_{t,\sigma}(\nu) \in \dot{C}$).

In this case we fix such a t and set $F(\varrho) = f_{t,\sigma}(\nu)$ for a ν as in (d). Otherwise, we set $F(\varrho) = 0$.

Now, let $E(F)$ be the club of all ε such that $\mathrm{rank}(\varepsilon, \varepsilon) > \tau$ and $F``\varepsilon \subseteq \varepsilon$. Note that F and $E(F)$ are definable from parameters in $M^*[G]$, and therefore they are in $M^*[G]$ as well. Moreover, by the $\aleph_2$–c.c., we may assume that $E(F)$ is in $M[G]$. Using the induction hypothesis, we know that there is some $\varepsilon \in \delta_M$ which is a limit point of $E(F)$ and such that $g_{\delta_M}(\varepsilon) = n$. Let $\varrho < \eta < \varepsilon$ be such that $\mathrm{rank}(\varrho, \varrho) \geq \delta + 1$, $\eta \in E(F)$, $g_{\delta_M}``[\varrho, \eta) = \{n\}$, and $[\varrho, \eta) \cap A_i = \emptyset$ for every $i \in \mathrm{dom}(b_{q',\sigma})$. These ordinals can be found since g_{δ_M} is continuous and since ε is a limit of ordinals in $E(F)$ and $\mathrm{rank}(\varepsilon, \varepsilon) > \tau$, together with the fact that $\mathrm{rank}(A_i, \varepsilon) \leq \tau$ for every $i \in \mathrm{dom}(b_{q',\sigma})$. Let $t \in \mathcal{P}_\alpha \in M^*[G]$ witness (a)–(d) for ϱ. The existence of such a t follows from the fact that q'' witnesses (a)–(d) in $\mathbf{V}[G]$, where $\mathrm{dom}(F_{q''}) = \mathrm{dom}(F_{q'})$ and $\Delta_{q''} = \Delta_{q'}$, $F_{q''}(\xi) = F_{q'}(\xi)$ for all $\xi \in \mathrm{dom}(F_{q'})$, $\xi \neq \sigma$, and where $F_{q''}(\sigma) = (f_{q',\sigma} \cup \{\langle \delta + 1, \varrho \rangle\}, b_{q',\sigma})$. It follows from (d) for t that there is some $\nu \in \mathrm{dom}(f_{t,\sigma})$ such that $\nu > \delta + 1$ and such that t forces $f_{t,\sigma}(\nu) \in \dot{C}$. Since we may assume that $f_{t,\sigma}(\nu) \in [\varrho, \eta)$ by the definition of $E(F)$ and the fact that $\eta \in E(F)$, we have that $g_{\delta_M}(f_{t,\sigma}(\nu)) = n$. By extending $q'|_\sigma$ if necessary we may assume that $q'|_\sigma$ decides the above facts holding in $\mathbf{V}[G]$. Finally, by the choice of ϱ such that $[\varrho, \eta) \cap A_i = \emptyset$ for all $i \in \mathrm{dom}(b_{q',\sigma})$, we may amalgamate q' and t into a $\mathcal{P}_\alpha$–condition $q^\dagger$. Since $g_{\delta_M}(f_{t,\sigma}(\nu)) = n$, $q^\dagger$ forces $f_{t,\sigma}(\nu) \in \dot{C}$, and q' was an arbitrary extension of q, we are done in this case.

Now assume that α is a limit ordinal. Let $q' \in \mathcal{P}_\alpha$ be an extension of q and let $\sigma \in M$ be an ordinal with $\sigma > \max(\mathrm{dom}(F_{q'}) \cap M)$. Let $G \subseteq \mathcal{P}_\sigma$ be a generic filter with $q'|_\sigma \in G$. By Lemma 3.7, G is also generic over M^*.

Note that the set D of all $\delta < \omega_1$ for which there is some $N \in \mathcal{M}_{\alpha+1}$ such that

(e) $\delta = \delta_N$,
(f) $\dot{C} \in N$, and
(g) $(N, \alpha) \in \Delta_t$ for some $t \in \mathcal{P}_\alpha$ such that $t|_\sigma \in G$ and $\mathrm{dom}(F_t) \subseteq \sigma$

is definable in $M^*[G]$.

Note also that D is cofinal in ω_1 since $\delta_M \in D$ as witnessed by $q|_\sigma$. Hence, if C is the set of the accumulation points of D, then C is a club definable in $M^*[G]$. Moreover, by the $\aleph_2$-c.c., we may assume $C \in M[G]$. Using the induction hypothesis, we know that there is some

$\varepsilon \in C \cap \delta_M$ such that $g_{\delta_M}(\varepsilon) = n$. Take $\eta < \varepsilon$ such that g_δ"$[\eta, \varepsilon) = \{n\}$. We may now find $\delta \in (\eta, \varepsilon) \cap D$. Let now $N \in \mathcal{M}_{\alpha+1}$ and t satisfy (e)–(g) for δ, and note that t forces $\delta = \delta_N \in \dot{C}$ by Lemma 3.7. As in the previous case, by extending $q'|_\sigma$ if necessary we may assume that $q'|_\sigma$ forces the above facts holding in $\mathbf{V}[G]$. Since t forces $\delta = \delta_N \in \dot{C}$ by Lemma 3.7, it suffices to find a common extension $q^\dagger$ of q' and t. But q' and t can indeed be amalgamated into an extension $q^\dagger$ by (g) for t. Since q' was an arbitrary extension of q, we are done in this case too.[7] □

We will next prove Theorem 1.16. We start by defining a natural forcing for destroying a potential $\mho$–sequence (cf. [6] and [7]).

Definition 3.10. Let $\mathcal{G} = \langle g_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ be such that each g_δ is a continuous function from δ into ω with respect to the order topology. Let $\mathbb{B}_\mathcal{G}$ be the forcing notion consisting of all pairs $(f, \langle b_\xi : \xi \in D\rangle)$ satisfying the following conditions.

(1) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function such that $\mathrm{rank}(f(\nu), f(\nu)) \geq \nu$ for every $\nu \in \mathrm{dom}(f)$.
(2) b is a function with $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and such that for each ξ in $\mathrm{dom}(b)$,
  (2.1) $b(\xi) \in \omega$,
  (2.2) $g_{f(\xi)}$" $\mathrm{range}(f) \subseteq \omega \setminus \{b(\xi)\}$, and
  (2.3) $\mathrm{rank}(\{\gamma < f(\xi) \mid g_{f(\xi)}(\gamma) \neq b(\xi)\}, f(\xi)) = \mathrm{rank}(f(\xi), f(\xi))$.

Given $\mathbb{B}_\mathcal{G}$–conditions $c^\epsilon = (f^\epsilon, b^\epsilon)$ for $\epsilon \in \{0, 1\}$, c^1 extends c^0 if and only if $f^0 \subseteq f^1$, and $b^0 \subseteq b^1$.

The forcing $\mathcal{P}$ witnessing Theorem 1.16 is $(\mathcal{P}_\kappa, \leq_\kappa)$, where the sequence $\langle(\mathcal{P}_\alpha, \leq_\alpha) \mid \alpha \leq \kappa\rangle$ will be defined soon by recursion. The sequence $\langle(\mathcal{P}_\alpha, \leq_\alpha) \mid \alpha \leq \kappa\rangle$ will be a finite support iteration with symmetric systems as side conditions, where the members of the side conditions are of the form $(N, \vec{\mathcal{W}}, \gamma)$, with $N \preccurlyeq H(\kappa)$ countable, $\vec{\mathcal{W}}$ an $\subseteq$–decreasing ω–sequence of collections of ordered pairs $(M, \mathcal{Z})$ where, again, $M \preccurlyeq H(\kappa)$ is countable, and $\gamma < \kappa$, rather than just pairs (N, γ), for N and γ as above, as in the previous construction. The role of the sequence of predicates $\vec{\mathcal{W}}$ will be to guide the inductive argument in the proof of properness (Lemma 3.18).

To be a bit more specific, we will be given a strong WCG–sequence $\langle A_\delta : \delta < \mathrm{Lim}(\omega_1)\rangle$ in the ground model that we will want to preserve

[7]The proof in the successor case could be simplified a bit by looking, in the definition of F, at conditions t such that $\sigma \notin \mathrm{dom}(F_t)$ (similarly as what we do in the limit case).

while killing $\mho$ in the end. At a typical successor stage $\alpha = \sigma + 1$ of the proof of Lemma 3.18, if $\dot{g}$ is the name for a potential witness to $\mho$ picked by our book–keeping that we intend to kill at that stage and if $(N, (\mathcal{W}_n)_{n<\omega}, \alpha)$ is a relevant triple for a condition q at that stage, then for the inductive proof to go through we will need that $q|_\sigma$ forces that for each n and each final segment A' of A_{δ_N}, the set of relevant triples $(M, \mathcal{Z}, \sigma)$ coming from some condition in $\dot{G}_\sigma$ such that $\delta_M \in A'$ and $\dot{g}_{\delta_N}(\delta_M) \neq m$, where m is the colour we have committed ourselves to avoid on A_{δ_N}, is large in some suitable sense. One important point is that, by the induction hypothesis, we will be able to find extensions of $q|_\sigma$ which are $(M, \mathcal{P}_\sigma)$–generic for such M's.

Definition 3.11. Let N and $\mathcal{U}$ be two sets. We will say that $\mathcal{U}$ is N*–unbounded* in case for every $x \in N$ there is some $M \in \mathcal{U} \cap N$ such that $x \in M$.

Given a set N, let us say that N *is sufficiently closed* if for every finite set x, $x \subseteq N$ iff $x \in N$.

Remark 3.12. If $\mathcal{U}$ is N–unbounded, $\mathcal{U}' \subseteq \mathcal{U}$ and every member of $\{N\} \cup \mathcal{U}$ is sufficiently closed, then at least one of $\mathcal{U}'$, $\mathcal{U} \backslash \mathcal{U}'$ is N–unbounded.

Given an ω–sequence $\vec{\mathcal{U}}$, we will use $\mathcal{U}_n$ to denote the nth member of $\vec{\mathcal{U}}$; that is, $\vec{\mathcal{U}} = (\mathcal{U}_n)_{n<\omega}$. Given two ω–sequences $\vec{\mathcal{U}}$, $\vec{\mathcal{V}}$ of sets, by $\vec{\mathcal{V}} \subseteq \vec{\mathcal{U}}$ we will mean that $\mathcal{V}_n \subseteq \mathcal{U}_n$ for all n. Finally, an ω–sequence $\vec{\mathcal{U}}$ is $\subseteq$*–decreasing* if $\mathcal{U}_m \subseteq \mathcal{U}_n$ whenever $n < m < \omega$.

Let $\mathcal{A} = \langle A_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ be a strong WCG–sequence. For every $\delta \in \mathrm{Lim}(\omega_1)$, let $(A_\delta(m))_{m<\omega}$ be the strictly increasing enumeration of A_δ and define $A^\delta_m := A_\delta \backslash A_\delta(m)$.

Lemma 3.13. *Let $\mathcal{G} = \langle g_\delta \mid \delta \in \omega_1\rangle$ be such that each g_δ is a continuous function from δ into ω and let us denote by $\mathcal{M}$ the set of all sufficiently closed countable subsets N of $H(\kappa)$ such that $N \models \mathrm{ZFC}^*$, δ_N exists and $\omega_1^N = \omega_1$. Suppose $p < \omega$ and $\{N_i : i < p\}$, $\{\vec{\mathcal{U}}_i : i < p\}$ are such that for all $i < p$, $n < \omega$ and $m < \omega$, the following hold.*

($\circ$) *$N_i \in \mathcal{M}$.*
($\circ$) *$\delta_{N_{i'}} = \delta_{N_i}$ for all i'.*
($\circ$) *$\vec{\mathcal{U}}_i$ is a $\subseteq$–decreasing ω–sequence.*
($\circ$) *$\mathcal{U}_{i,n} \subseteq N_i \cap \mathcal{M}$.*
($\circ$) *$\{M \in \mathcal{U}_{i,n} \mid \delta_M \in A^{\delta_{N_i}}_m\}$ is N_i–unbounded.*

Then, for each s in $[\omega]^{p+1}$ and each $i < p$ there is some $s_i \in [s]^p$ such that

$$\{M \in \mathcal{U}_{i,n} \mid \delta_M \in A^{\delta_{N_i}}_m \text{ and } g_{\delta_{N_i}}(\delta_M) \notin s_i\}$$

is N_i–unbounded for all n, $m < \omega$. In particular, for each s in $[\omega]^{p+1}$ there is at least one colour $j \in s$ such that

$$\{M \in \mathcal{U}_{i,n} \mid \delta_M \in A_m^{\delta_{N_i}} \textit{ and } g_{\delta_{N_i}}(\delta_M) \neq j\}$$

is N_i–unbounded for all $i < p$, $n < \omega$ and $m < \omega$.

Proof. It suffices to note that for each pair $\{n_0, n_1\} \in [s]^2$, the above remark (together with the fact that $\vec{\mathcal{U}}_i$ is a $\subseteq$–decreasing ω–sequence) ensures the existence of $j \in 2$ such that

$$\{M \in \mathcal{U}_{i,n} \mid \delta_M \in A_m^{\delta_{N_i}} \text{ and } g_{\delta_{N_i}}(\delta_M) \neq n_j\}$$

is N_i–unbounded for co–finally many n, $m < \omega$, and therefore for all n, $m < \omega$. $\square$

Notation 3.14. If $\mathcal{W}$ is a set of ordered pairs, then the projection $\{a \mid \langle a, b\rangle \in \mathcal{W}$ for some $b\}$ will be denoted by $\pi_0(\mathcal{W})$. Similarly, $\pi_1(\mathcal{W})$ denotes the projection $\{b \mid \langle a, b\rangle \in \mathcal{W}$ for some $a\}$

Let us fix Φ and θ_α, $\mathcal{M}^*_\alpha$ and $\mathcal{M}_\alpha$ (for $\alpha \leq \kappa$) exactly as in the proof of Theorem 1.15. We proceed to the definition of $\langle \mathcal{P}_\alpha \mid \alpha \leq \kappa\rangle$ now. $\mathcal{P}_0$ consist of all pairs of the form $(\emptyset, \{(N_i, \vec{\mathcal{W}}_i, 0) \mid i < m\})$ with the following properties.

- $(a\,1)$ $\{N_i \mid i < m\}$ is a finite Φ–symmetric system.
- $(a\,2)$ For every $i < m$, $\vec{\mathcal{W}}_i = (\mathcal{W}_{i,n})_{n<\omega}$ is a $\subseteq$–decreasing ω–sequence such that each $\mathcal{W}_{i,n}$ is a set of ordered pairs in N_i.
- $(a\,3)$ For every $i < m$ and $n < \omega$, every member of $\pi_0(\mathcal{W}_{i,n})$ is a countable $M \preccurlyeq (H(\kappa), \in, \Phi)$.

Given $\mathcal{P}_0$–conditions q and p, $q \leq_0 p$ iff $\Delta_p \subseteq \Delta_q$.

Let $\alpha = \sigma + 1$. Suppose $\Phi(\sigma)$ is a $\mathcal{P}_\sigma$–name for a c.c.c. poset or there is a $\mathcal{P}_\sigma$–name $\dot{\mathcal{G}}$ for a sequence of continuous functions $g_\delta : \delta \longrightarrow \omega$ ($\delta \in Lim(\omega_1)$) such that $\Phi(\sigma)$ is a $\mathcal{P}_\sigma$–name for $\mathbb{B}_{\dot{\mathcal{G}}}$. Then we let $\dot{\mathcal{Q}}_\sigma = \Phi(\sigma)$. In the other case we let $\Phi(\sigma)$ be a $\mathcal{P}_\sigma$–name for, say, trivial forcing on $\{0\}$.

If $\alpha = \sigma + 1$ and $\vec{\mathcal{W}} \in V$ is an ω–sequence whose members are sets of ordered pairs, then $\vec{\mathcal{W}}^{\dot{G}_\sigma}$ denotes the canonical $\mathcal{P}_\sigma$–name for the ω–sequence whose nth member is defined as

$$\mathcal{W}_n^{\dot{G}_\sigma} := \{(M, \vec{\mathcal{Z}}) \in \mathcal{W}_n \mid (M, \vec{\mathcal{Z}}, \sigma + 1) \in \Delta_u \text{ for some } u \in \dot{G}_\sigma\}$$

Note that if $\vec{\mathcal{W}}$ is $\subseteq$–decreasing, then $\mathcal{P}_\sigma$ forces that so is $\vec{\mathcal{W}}^{\dot{G}_\sigma}$.

If $\alpha \leq \kappa$ (regardless of whether α is a successor or a nonzero limit ordinal), the definition of $\mathcal{P}_\alpha$ is as follows. Conditions in $\mathcal{P}_\alpha$ are pairs of the form

$$q = (F_q,\, \Delta_q)$$

with the following properties.

$(b\,0)$ F_q is a finite function with $\mathrm{dom}(F_q) \subseteq \alpha$.

$(b\,1)$ Δ_q is of the form $\{(N_i, \vec{\mathcal{W}}_i, \beta_i) \mid i < m\}$ where, for all $i < m$, $\beta_i \leq \alpha \cap \sup(N_i \cap \kappa)$.

$(b\,2)$ For all $\xi < \alpha$, the restriction of q to ξ is a condition in $\mathcal{P}_\xi$.

$(b\,3)$ If $\xi \in \mathrm{dom}(F_q)$, then $q|_\xi$ forces (in $\mathcal{P}_\xi$) that $F_q(\xi) \in \dot{\mathcal{Q}}_\xi$.

$(b\,4)$ Suppose $\xi \in \mathrm{dom}(F_q)$ and there is some $\mathcal{P}_\xi$–name $\dot{\mathcal{G}}$ for a sequence of continuous functions $g_\delta : \delta \longrightarrow \omega$ (for $\delta \in Lim(\omega_1^{\mathbf{V}})$) such that $\Phi(\xi)$ is a $\mathcal{P}_\xi$–name for $\mathbb{B}_{\dot{\mathcal{G}}}$. Suppose $(N, \vec{\mathcal{W}}, \beta) \in \Delta_q$, $\beta \geq \xi + 1$, $N \in \mathcal{M}_{\xi+1}$, and

$$\{M \in \pi_0(\mathcal{W}_n) \mid \delta_M \in A_m^{\delta_N}\}$$

is N–unbounded for all n, $m \in \omega$. Then δ_N is a fixed point of $f_{q,\xi}$, $b_{q,\xi}(\delta_N)$ is defined, and $q|_\xi$ forces that

$$\{M \in \pi_0(\mathcal{W}_n^{\dot{G}_\xi}) \mid \dot{g}_{\xi,\delta_N}(\delta_M) \neq b_{q,\xi}(\delta_N) \text{ and } \delta_M \in A_m^{\delta_N}\}$$

is N–unbounded for all $n, m \in \omega$.

Here, and throughout the proof of Theorem 1.16, we are using the obvious notational conventions when writing things like $f_{q,\xi}$ and $b_{q,\xi}$ (as in the proof of Theorem 1.15). Given conditions

$$q^\epsilon = (F_\epsilon,\ \{(N_i^\epsilon, \vec{\mathcal{W}}_i^\epsilon, \beta_i^\epsilon) \mid i < m_\epsilon\})$$

(for $\epsilon \in \{0, 1\}$) in $\mathcal{P}_\alpha$, we will say that $q^1 \leq_\alpha q^0$ if and only if the following holds.

$(c\,1)$ For all $\xi < \alpha$, $q^1|_\xi \leq_\xi q^0|_\xi$.

$(c\,2)$ $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and, for all $\xi \in \mathrm{dom}(F_0)$,

$$q^1|_\xi \Vdash_\xi F_1(\xi) \leq_{\dot{\mathcal{Q}}_\xi} F_0(\xi)$$

$(c\,3)$ For all $i < m_0$ there is some $\widetilde{\beta}_i \geq \beta_i^0$ such that $(N_i^0, \vec{\mathcal{W}}_i^0, \widetilde{\beta}_i) \in \Delta_{q^1}$.

As in the proof of Theorem 1.15, (i.e., as in [6]) we can show that the corresponding versions of Lemmas 3.3, 3.4 and 3.5 hold for our present iteration. The proof of the corresponding form of Lemma 3.7 will need to be a bit more elaborate.

Definition 3.15. Given an ordinal $\alpha \leq \kappa$ and a pair $(N, \vec{\mathcal{W}})$ such that $N = N^* \cap H(\kappa)$ for some $N^* \in \mathcal{M}_\alpha^*$ and such that $\vec{\mathcal{W}}$ is a $\subseteq$–decreasing ω–sequence of sets consisting of ordered pairs in N, we will say that $\vec{\mathcal{W}}$ *is N–α–large* whenever the following conditions hold.

(1) For all $n < \omega$, every member of $\pi_0(\mathcal{W}_n)$ is a countable elementary substructure of $(H(\kappa), \in, \Phi)$. If $\alpha = 0$, we also require that $\pi_1(\mathcal{W}_n) = \{\emptyset\}$
(2) For all n, $m < \omega$, $\{M \in \pi_0(\mathcal{W}_n) \mid \delta_M \in A_m^{\delta_N}\}$ is N–unbounded.
(3) For every nonzero ordinal $\beta \in N \cap \alpha$ there is some $n < \omega$ with the property that for every $(M, \vec{\mathcal{W}}') \in \mathcal{W}_n$,
  - ($\circ$) $M = M^* \cap H(\kappa)$ for some $M^* \in \mathcal{M}^*_\beta$ with $\mathcal{P}_\beta \in M^*$ and
  - ($\circ$) $\vec{\mathcal{W}}'$ is M–β–large.

Lemma 3.16. *Let $\alpha \leq \kappa$ and let $N^* \in \mathcal{M}^*_\alpha$ be such that $\mathcal{A} \in N^*$ and $(\mathcal{P}_\beta)_{\beta<\alpha} \in N^*$. Let $N = N^* \cap H(\kappa)$. Then there is a $\subseteq$–decreasing ω–sequence $\vec{\mathcal{W}}$ such that $\vec{\mathcal{W}}$ is N–α–large.*

Proof. The proof is by induction on α. We first show that for every $m \in \omega$, every x in N and every finite set $s \subseteq N \cap \alpha$ there is some $(M, \vec{\mathcal{W}}') \in N$ such that $M \preccurlyeq (H(\kappa), \in, \Phi)$, $|M| = \aleph_0$, $M \in \mathcal{M}_{\max(s)}$ if $s \neq \emptyset$, $x \in M$, and such that $\delta_M \in A_m^{\delta_N}$. If s is empty, we simply define $\vec{\mathcal{W}}'$ as the empty set. Otherwise, we also require that $\vec{\mathcal{W}}'$ be M–max(s)–large and $M = M^* \cap H(\kappa)$ for some $M^* \in \mathcal{M}^*_{\max(s)}$ with $\{s, \mathcal{P}_{\max(s)}\} \subseteq M^*$.

If $s \neq \emptyset$, let $C \in N^*$ be a club of ω_1 such that for every $\delta \in C$ there is some $M^* \in \mathcal{M}_{\max(s)}$ such that $\{x, s, \mathcal{P}_{\max(s)}\} \subseteq M^*$ and $\delta_{M^*} = \delta$. (If $s = \emptyset$, we take $C \in N^*$ to be a club of ω_1 such that for every $\delta \in C$ there is some countable $M \preccurlyeq (H(\kappa), \in, \Phi)$ such that $x \in M$ and $\delta_M = \delta$.) Since $\mathcal{A} \in N^*$, by Fact 1.6, we may fix $\delta \in C \cap A_m^{\delta_N}$. By correctness of N^* there is then some $M^* \in \mathcal{M}_{\max(s)} \cap N^*$ such that $\{x, s, \mathcal{P}_{\max(s)}\} \subseteq M^*$ and such that $\delta_{M^*} = \delta$, if $s \neq \emptyset$, and a countable $M \preccurlyeq (H(\kappa), \in, \Phi)$, $M \in N^*$, such that $x \in M$ and $\delta_M = \delta$ if $s = \emptyset$. Let $M = M^* \cap H(\kappa)$ (if $s \neq \emptyset$). By induction hypothesis there is some ω–sequence $\vec{\mathcal{W}}'$ such that $\vec{\mathcal{W}}'$ is M–max(s)–large if $s \neq \emptyset$, and by correctness of N^* there is some such $\vec{\mathcal{W}}'$ in N. Now $(M, \vec{\mathcal{W}}')$ is as desired.

For every $m < \omega$, $x \in N$, and every finite subset of $N \cap \alpha$, choose a pair $(M, \vec{\mathcal{W}}')$ as above and let $\mathcal{W}_0$ be the set of all these pairs. If $\alpha = 0$, we let $\mathcal{W}_{n+1} = \mathcal{W}_0$ for every n. If $\alpha > 0$, we fix a surjection $f : \omega \longrightarrow N \cap \alpha$. In this case, $\mathcal{W}_{n+1}$ is defined as the set of those pairs $(M, \vec{\mathcal{W}}')$ in $\mathcal{W}_0$ such that, letting $\mu = \max(\text{range}(f \restriction n+1))$, $\vec{\mathcal{W}}'$ is M–μ–large and $M = M^* \cap H(\kappa)$ for some $M^* \in \mathcal{M}^*_\mu$ with $\{\text{range}(f \restriction n+1), \mathcal{P}_\mu\} \subseteq M^*$. Since $\vec{\mathcal{W}} = (\mathcal{W}_n)_{n<\omega}$ is $\subseteq$–decreasing, it is clear that this ω–sequence is N–α–large. $\square$

We consider the following natural notion of pre–properness (cf. Definition 3.6).

Definition 3.17. Given $\alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a pair $(N, \vec{\mathcal{W}})$, we say that q *is $\mathcal{P}_\alpha$–pre-generic* for $(N, \vec{\mathcal{W}})$ in case

- $\alpha < \kappa$ and $(N, \vec{\mathcal{W}}, \alpha) \in \Delta_q$, or else
- $\alpha = \kappa$ and $(N, \vec{\mathcal{W}}, \sup(N \cap \kappa)) \in \Delta_q$.

Our properness lemma now is the following.

Lemma 3.18. *Suppose $\alpha \leq \kappa$ and $N^* \in \mathcal{M}^*_\alpha$. Let $N = N^* \cap H(\kappa)$ and assume that $\vec{\mathcal{W}}$ is a $\subseteq$–decreasing ω–sequence consisting of ordered pairs in N and such that for all $n < \omega$, every member of $\pi_0(\mathcal{W}_n)$ is a countable elementary substructure of $(H(\kappa), \in, \Phi)$. Then the following conditions hold.*

$(1)_\alpha$ *For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is $\mathcal{P}_\alpha$–pre-generic for $(N, \vec{\mathcal{W}})$.*

$(2)_\alpha$ *If*
- $(\circ)$ *$\mathcal{P}_\alpha \in N^*$ and $\vec{\mathcal{W}}$ is N–α–large and*
- $(\circ)$ *$q \in \mathcal{P}_\alpha$ is $\mathcal{P}_\alpha$–pre-generic for $(N, \vec{\mathcal{W}})$,*

then q is $(N^*, \mathcal{P}_\alpha)$–generic.

$(3)_\alpha$ *For all $q \in \mathcal{P}_\alpha$, m^*, $n^* < \omega$, and $x \in N$, if*
- $(\circ)$ *q is $\mathcal{P}_\alpha$–pre-generic for $(N, \vec{\mathcal{W}})$ and*
- $(\circ)$ *$\{M \in \pi_0(\mathcal{W}_n) \mid \delta_M \in A^{\delta_N}_m\}$ is N–unbounded for all m, $n < \omega$,*

then there is some condition $q' \in \mathcal{P}_\alpha$ extending q and some $(M, \vec{\mathcal{W}}') \in \mathcal{W}_{n^*}$ such that
- (a) *$x \in M$,*
- (b) *$\delta_M \in A^{\delta_N}_{m^*}$, and*
- (c) *q' is $\mathcal{P}_\alpha$–pre-generic for $(M, \vec{\mathcal{W}}')$.*

Proof. The proof will be by induction on α. The case $\alpha = 0$ follows from Corollary 2.3 (the proof of $(3)_0$ is like the proof of that corollary, using the fact that $\{M \in \pi_0(\mathcal{W}_n) \,:\, \delta_M \in A^{\delta_N}_\nu\}$ is N–unbounded for all $m < \omega$ and $n < \omega$).

Now we prove the conclusion for the case $\alpha = \sigma + 1$. We may assume that there is some $\mathcal{P}_\sigma$–name $\dot{\mathcal{G}}$ for a sequence of continuous functions $g_\delta : \delta \longrightarrow \omega$ (for $\delta \in \mathrm{Lim}(\omega_1^{\mathbf{V}})$) such that $\Phi(\sigma)$ is a $\mathcal{P}_\sigma$–name for $\mathbb{B}_{\dot{\mathcal{G}}}$ (otherwise the proof is easier). Let also $\dot{g}_\delta$ be a name for g_δ for every δ as above. We start with the proof of $(1)_\alpha$. By $(1)_\sigma$ we may assume, by extending $q|_\sigma$ if necessary, that $q|_\sigma$ is $\mathcal{P}_\sigma$–pre-generic for $(N, \vec{\mathcal{W}})$. If the hypothesis of $(b\,4)$ fails for q and σ, then

$(b\,4)$ holds vacuously for this pair. So, assume that $\sigma \in \mathrm{dom}(F_q)$ and $\{M \in \pi_0(\mathcal{W}_n) : \delta_M \in A_m^{\delta_N}\}$ is N–unbounded for all n, $m \in \omega$. By $(3)_\sigma$, $q|_\sigma$ forces that $\{M \in \pi_0(\mathcal{W}_n^{\dot{G}_\sigma}) \mid \delta_M \in A_m^{\delta_N}\}$ is N–unbounded for all $n, m \in \omega$. By Lemma 3.13, we may assume, by extending $q|_\sigma$ if necessary, that there is a colour $j \in \omega$ such that $q|_\sigma$ forces that j is not in the range of $\dot{g}_{\sigma,\delta_N} \restriction \mathrm{range}(f_{q,\sigma})$ and

$$\{M \in \pi_0(\mathcal{W}_n^{\dot{G}_\sigma}) \mid \dot{g}_{\sigma,\delta_N}(\delta_M) \neq j \text{ and } \delta_M \in A_m^{\delta_N}\}$$

is N–unbounded for all $n, m \in \omega$. In this case, it suffices to define q' in such a way that $\Delta_{q'} = \Delta_q \cup \{(N, \vec{\mathcal{W}}, \alpha)\}$, $F_{q'} \restriction \sigma = F_q \restriction \sigma$, $f_{q',\sigma} = f_{q,\sigma} \cup \{(\delta_N, \delta_N)\}$ and $b_{q',\sigma} = b_{q,\sigma} \cup \{(\delta_N, j)\}$.

Given $q \in \mathcal{P}_\kappa$, we will say that a countable structure M is *relevant for q at stage ξ* if there are some $\vec{\mathcal{W}}$ and β such that $(M, \vec{\mathcal{W}}, \beta) \in \Delta_q$ $\beta \geq \xi + 1$, $N \in \mathcal{M}_{\xi+1}$ and $\{M \in \pi_0(\mathcal{W}_n) \mid \delta_M \in A_m^{\delta_N}\}$ is N–unbounded for all $n, m \in \omega$. The following claim can be proved by essentially repeating the same argument as above (based on $(3)_\sigma$ together with Lemma 3.13) finitely many times.

Claim 3.19. *The set of $q \in \mathcal{P}_\alpha$ such that $\sigma \in \mathrm{dom}(F_q)$ is a dense subset of $\mathcal{P}_\alpha$. Moreover, if $q \in \mathcal{P}_\alpha$, $\sigma \notin \mathrm{dom}(F_q)$, $q|_\alpha$ forces that (f, b) is in $\dot{\mathcal{Q}}_\sigma$, and* $\max(\mathrm{range}(f)) < \min\{\delta_M \mid M$ *is relevant at stage* $\sigma\}$, *then there is a condition r extending q such that $\sigma \in \mathrm{dom}(F_r)$, $f \subseteq f_{r,\sigma}$ and $b \subseteq b_{r,\sigma}$.*

Let us proceed to the proof of $(3)_\alpha$ now. By the above claim, we may assume that $\sigma \in \mathrm{dom}(F_q)$. In particular, δ_N is a fixed point of $f_{q,\sigma}$. Since $\dot{\mathcal{G}}$ is forced to be a sequence of continuous functions, we may assume, by extending $q|_\sigma$ if necessary, that there is an ordinal $\eta < \delta_N$ such that $q|_\sigma$ forces that for every $\mu \in [\eta, \delta_N]$ and every $\nu \in \mathrm{dom}(b_{q,\sigma}) \setminus (\delta_N + 1)$, $\dot{g}_\nu(\mu) \neq b_{q,\sigma}(\nu)$. By condition $(b\,4)$ in the definition of $\mathcal{P}_\alpha$ and by extending $q|_\sigma$ if necessary, we can find some $(M, \vec{\mathcal{W}}') \in \mathcal{W}_{n^*}$ such that $\{\eta, x, f_{q,\sigma} \restriction \delta_N\} \subseteq M$, $\delta_M \in A_{m^*}^{\delta_N}$, $(M, \vec{\mathcal{W}}', \sigma) \in \Delta_{q|_\sigma}$, and such that $q|_\sigma$ forces that $\dot{g}_{\delta_N}(\delta_M) \neq b_{q,\sigma}(\delta_N)$. Hence, $q|_\sigma$ forces that for all $\nu \in \mathrm{dom}(b_{q,\sigma}) \setminus \delta_M$, $\dot{g}_\nu(\delta_M) \neq b_{q,\sigma}(\nu)$. If $M \notin \mathcal{M}_\alpha$ or $\{M' \in \pi_0(\mathcal{W}_n') \mid \delta_M \in A_m^{\delta_M}\}$ fails to be M–unbounded for any $m < \omega$ and $n < \omega$, then the pair

$$q' = (F_q, \Delta_q \cup \{(M, \vec{\mathcal{W}}', \alpha)\})$$

is a condition in $\mathcal{P}_\alpha$ extending q, so we are done in this case. Otherwise, we can find q' using $(2)_\sigma$ together with Lemma 3.13 (in the extension via $\mathcal{P}_\sigma$) as in the relevant case of the proof of $(1)_\alpha$.

Now let us prove $(2)_\alpha$. Let E be a maximal antichain of $\mathcal{P}_\alpha$ in N^*. By extending q if necessary we may assume that q extends a condition in E. We want to show that such a condition is in N and for this it will suffice to find a member of $E \cap N$ compatible with q. Let G_σ be a $\mathcal{P}_\sigma$–generic filter over V with $q|_\sigma \in G_\sigma$. By $(2)_\sigma$ we have that G_σ is also generic over N^*.

By Claim 3.19 we may assume that $\sigma \in \mathrm{dom}(F_q)$. In particular, δ_N is a fixed point of $f_{q,\sigma}$. As in the proof of $(3)_\alpha$, we may assume, by extending $q|_\sigma$ if necessary, that there is an ordinal $\eta < \delta_N$ such that $q|_\sigma$ forces that for every $\mu \in [\eta, \delta_N]$ and every $\nu \in \mathrm{dom}(b_{q,\sigma}) \setminus (\delta_N + 1)$, $\dot{g}_\nu(\mu) \neq b_{q,\sigma}(\nu)$. Let $\rho = \max(\mathrm{range}(f_{q,\sigma} \restriction \delta_N))$ and define E^+ as the (partially defined) function sending each $p \in E$ to the Φ–first $\mathcal{P}_\alpha$–condition p^+ such that $p^+|_\sigma \in G_\sigma$, $f_{p,\sigma} \restriction \delta_N \subseteq f_{p^+,\sigma}$, $\mathrm{range}(f_{p,\sigma}) \cap \rho = \mathrm{range}(f_{p^+,\sigma}) \cap \rho$ and $b_{p,\sigma} \restriction \delta_N \subseteq b_{p^+,\sigma}$. By the $\aleph_2$–c.c., E^+ is not only in $N^*[G_\sigma]$, but in $N[G_\sigma]$ (i.e., we can find in N a $\mathcal{P}_\alpha$–name for E^+). Since $\vec{\mathcal{W}}$ is N–α–large, by condition $(b\,4)$ we can find $n \in \omega$ and $(M, \vec{\mathcal{W}}') \in \mathcal{W}_n$ such that

(i) $\vec{\mathcal{W}}'$ is M–σ–large and $M = M^* \cap H(\kappa)$ for some $M^* \in \mathcal{M}^*_\sigma$ such that $\mathcal{P}_\sigma \in M^*$,
(ii) $(M, \vec{\mathcal{W}}', \sigma) \in \Delta_u$ for some $u \in G_\sigma$,
(iii) $\dot{g}_{\delta_N}(\delta_M) \neq b_{q,\sigma}(\delta_N)$, and
(iv) $\{\eta, \rho, E^+\} \subseteq M[G_\sigma]$.

Since $q|_\sigma$ and u are both in G_σ, we may assume that $q|_\sigma$ extends u. We may also assume that there is an ordinal $\eta' \in (\eta,\, \delta_M)$ such that $q|_\sigma$ forces that for every ordinal $\mu \in [\eta', \delta_M]$, $\dot{g}_{\delta_N}(\mu) \neq b_{q,\sigma}(\delta_N)$. Now, in $M[G_\sigma]$, there is $r \in \mathrm{range}(E^+)$ such that $\min(\mathrm{range}(f_{r,\sigma}) \setminus \rho) > \eta'$. Note that $\max(\mathrm{range}(f_{r,\sigma})) < \delta_M$ since $\delta_M = \delta_{M[G_\sigma]}$ (this equality follows from the conjunction of (i), (ii) and $(2)_\sigma$). Since $r|_\sigma$ and $q|_\sigma$ are both in G_σ, we may assume, by extending $q|_\sigma$ if necessary, that $q|_\sigma$ decides r and extends its corresponding restriction. Finally note that the inequalities $\eta < \eta' < \min(\mathrm{range}(f_{r,\sigma}) \setminus \rho) \leq \max(\mathrm{range}(f_{r,\sigma})) < \delta_M$ ensure that $q|_\sigma$ forces that $F_q(\sigma)$ and $F_r(\sigma)$ are compatible in $\dot{\mathcal{Q}}_\sigma$. Hence, q and r are compatible.

It remains to prove the lemma for the case when α is a nonzero limit ordinal. The proof of $(1)_\alpha$ (resp., $(3)_\alpha$) is trivial using $(1)_\beta$ (resp., $(3)_\beta$) for all $\beta < \alpha$, together with the fact that $\mathrm{dom}(F_r)$ is bounded in α for any $\mathcal{P}_\alpha$–condition r. For $(2)_\alpha$, let $E \subseteq \mathcal{P}_\alpha$ be dense and open, $E \in N^*$, and let q satisfy the hypothesis of $(2)_\alpha$. We want to find a condition in $E \cap N^*$ compatible with q. We may assume that $q \in E$.

Suppose first that $\mathrm{cf}(\alpha) = \omega$. In this case we may take $\sigma \in N^* \cap \alpha$ above $\mathrm{dom}(F_q)$. Let G be $\mathcal{P}_\sigma$–generic with $q|_\sigma \in G$. In $N^*[G]$ it is true that there is a condition $q^\circ \in \mathcal{P}_\alpha$ such that

(a) $q^\circ \in E$ and $q^\circ|_\sigma \in G$, and
(b) $\mathrm{dom}(F_{q^\circ}) \subseteq \sigma$.

(the existence of such a q° is witnessed in $\mathbf{V}[G]$ by q.)

Since $q|_\sigma$ is $(N^*, \mathcal{P}_\sigma)$–generic by induction hypothesis, $q^\circ \in N^*$. By extending q below σ if necessary, we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. But now, if $q = (p, \Delta_q)$, the natural amalgamation $(p, \Delta_q \cup \Delta_{q^\circ})$ of q and q° is a $\mathcal{P}_\alpha$–condition extending them.

Finally, suppose $\mathrm{cf}(\alpha) \geq \omega_1$. We may assume that $\mathrm{dom}(F_q)$ is not bounded by $\sup(N \cap \alpha)$ as otherwise we can argue as in the $\mathrm{cf}(\alpha) = \omega$ case. The crucial observation now is that if $N' \in \mathrm{dom}(\Delta_q)$ and $\delta_{N'} < \delta_N$, then $\sup(N' \cap N \cap \alpha) \leq \sup(\Psi_{\overline{N},N}(N') \cap \alpha) \in N \cap \alpha$ whenever $\overline{N} \in \mathrm{dom}(\Delta_q)$ is such that $\delta_{\overline{N}} = \delta_N$ and $N' \in \overline{N}$. To see this, recall that $\Psi_{\overline{N},N}$ fixes $\overline{N} \cap N \cap \kappa$. Also, $\sup(\Psi_{\overline{N},N}(N') \cap \alpha) \in N \cap \alpha$ since $\Psi_{\overline{N},N}(N')$ is countable in N and α has uncountable cofinality. The symmetry of the systems $\mathrm{dom}(\Delta_q)$ is needed precisely to derive the conclusion that $\sup(N' \cap N \cap \alpha) < \sup(N \cap \alpha)$ for every $N' \in \mathrm{dom}(\Delta_q)$ with $\delta_{N'} < \delta_N$.

Hence we may fix $\sigma \in N \cap \alpha$ such that:

(i) $\sup(N' \cap N \cap \alpha) < \sigma$ for all $N' \in \mathrm{dom}(\Delta_q)$ with $\delta_{N'} < \delta_N$, and
(ii) if $\eta \in \mathrm{dom}(F_q)$ and $\eta < \sup(\alpha \cap N)$, then $\eta < \sigma$.

As in the case $\mathrm{cf}(\alpha) = \omega$, if G_σ is $\mathcal{P}_\sigma$–generic with $q|_\sigma \in G_\sigma$, then in $N^*[G_\sigma]$ we can find a condition $q^\circ \in \mathcal{P}_\alpha$ such that $q^\circ \in E$ and $q^\circ|_\sigma \in G_\sigma$, and such a q° will necessarily be in N^* by $(2)_\sigma$. By extending q below σ we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. The proof of $(2)_\alpha$ in this case will be finished if we can show that there is a condition $q^\dagger$ extending q and q°. The condition $q^\dagger$ can be built by recursion on $\mathrm{dom}(F_{q^\circ}) \setminus \sigma$ (note that $\sup(N \cap \alpha) \leq \min(\mathrm{dom}(F_q) \setminus \sigma)$ by the choice of σ, and therefore $\min(\mathrm{dom}(F_q) \setminus \sigma) > \max(\mathrm{dom}(F_{q^\circ}))$). The details of this construction, which for the sake of completeness we will sketch here, appear in [6].

Suppose $|\mathrm{dom}(F_{q^\circ}) \setminus \sigma| = n$ and let $\mu = \max(\mathrm{dom}(F_{q^\circ}))$. Applying Claim 3.19 n times, we get a condition $q^\triangle \in \mathcal{P}_{\mu+1}$ extending both $q|_{\mu+1}$ and $q^\circ|_{\mu+1}$ (note that if $\xi \in \mathrm{dom}(F_{q^\circ}) \setminus \sigma$ and M is relevant for q at stage ξ, then the choice of σ ensures that $\delta_M \geq \delta_N$). Finally note that if $\xi \in (\mu, \alpha) \cap \mathrm{dom}(F_q)$ and M is relevant either for q, or for q° or for $q^\triangle$ at stage ξ, then M must be relevant for q. (The choice of σ shows in fact that M cannot be relevant for q°. On the other hand, every M

associated to $q^{\triangle}$ is relevant at most at stage $\mu+1$ since $q^{\triangle} \in \mathcal{P}_{\mu+1}$.) Therefore,

$$q^{\dagger} = (F_{q^{\triangle}} \cup (F_q \restriction [\mu+1,\, \alpha)), \Delta_{q^{\triangle}} \cup \Delta_{q^{\circ}} \cup \Delta_q)$$

is a condition in $\mathcal{P}_\alpha$ extending both q and q°.

$\square$

The following corollary follows immediately from Lemmas 3.16 and 3.18.

Corollary 3.20. *$\mathcal{P}_\kappa$ is proper.*

Corollary 3.21. *$\mathcal{P}_\kappa$ forces that $\mathcal{A}$ is a* WCG*–sequence.*

Proof. Let $\dot{C}$ be a $\mathcal{P}_\kappa$–name for a club of ω_1 and let q be a $\mathcal{P}_\kappa$–condition. By the $\aleph_2$–c.c. of $\mathcal{P}_\kappa$ there is some $\alpha < \kappa$ such that $\dot{C}$ is a $\mathcal{P}_\alpha$–name and such that q is a condition in $\mathcal{P}_\alpha$. We may also assume that $\dot{C}$ is in $H(\kappa)$. Let us take a countable $N^* \preccurlyeq H(\theta)$ (for a large enough θ) containing everything relevant (this includes $\dot{C}$). Let $N = N^* \cap H(\kappa)$.

By Lemma 3.16 there is a $\subseteq$–decreasing ω–sequence $\vec{\mathcal{W}}$ such that $\vec{\mathcal{W}}$ is N–$(\alpha+1)$–large. By Lemma 3.18 $(1)_\alpha$ there is a condition q' in $\mathcal{P}_\alpha$ extending q and $\mathcal{P}_\alpha$–pre-generic for $(N, \vec{\mathcal{W}})$. It suffices to show that q' forces $\dot{C} \cap A^{\delta_N}_m \neq \emptyset$ for every $m \in \omega$.

Notice that the hypothesis of $(3)_\alpha$ in Lemma 3.18 is realized by q'. Hence, for every $n, m < \omega$, and every q'' extending q' there is some $\mathcal{P}_\alpha$–condition q^* extending q'' and there is some $(M, \vec{\mathcal{W}}')$ in $\mathcal{W}_n$ such that $\dot{C} \in M$, $\delta_M \in A^{\delta_N}_m$, and such that q^* is $\mathcal{P}_\alpha$–pre-generic for $(M, \vec{\mathcal{W}}')$. But by (3) in the definition of N–$(\alpha+1)$–large we may take n large enough so that $\vec{\mathcal{W}}'$ is necessarily M–α–large. Then, by Lemma 3.18 $(2)_\alpha$ we have that q^* is $(M^*, \mathcal{P}_\alpha)$–generic for any $M^* \in \mathcal{M}_\alpha$ containing $\mathcal{P}_\alpha$ such that $M^* \cap H(\kappa) = M$. But then q^* forces $\delta_M \in \dot{C}$. $\square$

Finally, by arguing as in the proof of Theorem 1.15 (i.e., by the usual book–keeping arguments using the $\aleph_2$–c.c. of $\mathcal{P}_\kappa$) we can prove that $\mathcal{P}_\kappa$ forces $\neg\mho$, MA and FA($\mathbb{B}$). This completes the proof of Theorem 1.16.

4. On polychromatic Ramsey theory

Polychromatic Ramsey theory[8] (see [9], [2] and [1]) is the study of colourings f of sets of the form $[X]^n$, focusing on the existence or non–existence of large sets $Y \subseteq X$ such that $c \restriction [Y]^n$ is a one–to–one function (such a set Y is usually called a *rainbow for c*). Here we consider the following negative polychromatic partition relation (see [1]).

[8]Also known as anti–Ramsey theory

Definition 4.1. $\omega_2 \not\to_{\text{poly}} (\omega_1)^2_{2\text{–bd}}$ means that there is a function f with the following properties.

(a) $\mathrm{dom}(f) = [\omega_2]^2$
(b) f is 2–bounded (which means that $|f^{-1}(x)| \leq 2$ for every x).
(c) f does not have rainbows of order–type ω_1 (i.e., there is no $Y \subseteq \omega_2$ of order type ω_1 such that the restriction of f to $[Y]^2$ is one to one).

In [1], Abraham and Cummings show the consistency of the relation $\omega_2 \not\to_{\text{poly}} (\omega_1)^2_{2\text{–bd}}$. In their model, $2^{\aleph_0} = \aleph_2$. One of the questions asked in [1] is whether $\omega_2 \not\to_{\text{poly}} (\omega_1)^2_{2\text{–bd}}$ is consistent together with $2^{\aleph_0} > \aleph_2$. Here we give an affirmative answer to this question. In fact, we show that if CH holds and λ is any cardinal, then there is a proper poset with the $\aleph_2$–c.c. forcing $\omega_2 \not\to_{\text{poly}} (\omega_1)^2_{2\text{–bd}}$ together with $\mathrm{FA}(\mathbb{B})_\lambda$. Our result is the following.

Theorem 4.2. *(CH) If κ is a regular cardinal such that $2^{<\kappa} = \kappa$, then there exists a proper forcing notion $\mathcal{P}$ with the $\aleph_2$–chain condition such that* FA($\mathbb{B}$)*, $2^\lambda = \kappa$ for every infinite $\lambda < \kappa$, and $\omega_2 \not\to_{poly} (\omega_1)^2_{2\text{–}bd}$ hold in the generic extension by $\mathcal{P}$.*

Starting from CH and using symmetric systems,[9] Abraham and Cummings generically add a function c such that

(1) $c : [\omega_2]^2 \longrightarrow \omega_1$,
(2) There are no $\alpha_0 < \alpha_1 < \alpha_2 < \beta < \omega_2$ such that $c(\alpha_0, \beta) = c(\alpha_1, \beta) = c(\alpha_2, \beta)$, and
(3) For every $X \subseteq \omega_2$ of order type ω_1 there are $\alpha_0 < \alpha_1 < \beta$ in X such that $c(\alpha_0, \beta) = c(\alpha_1, \beta)$.

It is straightforward to check that $f(\alpha, \beta) = (c(\alpha, \beta), \beta)$ witnesses $\omega_2 \not\to_{\text{poly}} (\omega_1)^2_{2\text{–bd}}$. Let us assume the hypotheses of Theorem 4.2 and let $\Phi : \kappa \longrightarrow H(\kappa)$ be a bijection. Let also $(e_\alpha)_{\alpha<\omega_2}$ be Φ–first such that each e_α is a bijection between α and $|\alpha|$. The following is (essentially) the forcing in [1] for adding c, which we will call $(\mathcal{P}_0, \leq_0)$.

Definition 4.3. Conditions in $\mathcal{P}_0$ are triples $p = (\emptyset, c_p, \Delta_p)$ with the following properties.

(1) c_p is a finite partial function from $[\omega_2]^2$ to ω_1.
(2) There are no $\alpha_0 < \alpha_1 < \alpha_2 < \beta$ in the domain of c_p such that $c_p(\alpha_0, \beta) = c_p(\alpha_1, \beta) = c_p(\alpha_2, \beta)$.
(3) For every $(\alpha, \beta) \in \mathrm{dom}(c_p)$, $c_p(\alpha, \beta) \geq e_\beta(\alpha)$.

[9] In our terminology.

(4) Δ_p is a finite set of pairs of the form $(N, 0)$ such that $\mathrm{dom}(\Delta_p)$ is a Φ–symmetric system.

Given p and q conditions in $\mathcal{P}_0$, $q \leq_0 p$ whenever the following holds:

(i) $c_p \subseteq c_q$,
(ii) $\Delta_p \subseteq \Delta_q$, and
(iii) for every $(\alpha, \beta) \in \mathrm{dom}(c_q)\setminus\mathrm{dom}(c_p)$ and every pair $(N, 0) \in \Delta_p$, if $\{\alpha, \beta\} \in N$, then $c_q(\alpha, \beta) \in N$.

In [1], what we have called here Δ_q is an actual Φ–symmetric system (i.e., there is no mention of 0's). Here we add the 'marker' 0 to the N's occurring in Δ_q – as well as the first vacuous component $\emptyset$, which of course does not show up in [1] either – since we are going to incorporate this forcing into an iteration with symmetric systems of structures and markers as in the other constructions in this paper. The restriction to 0 of any condition in $\mathcal{P}_\alpha$, for any $\alpha > 0$, will be a condition in $\mathcal{P}_0$ by definition, and this is the need for the $\emptyset$'s and 0's in Definition 4.3. Another difference between the original definition and the poset in Definition 4.3 is that the N's occurring in conditions in [1] are countable elementary substructures of $H(\omega_2)$ and not of $H(\kappa)$. These differences have no effect whatsoever on the proofs of the following lemmas from [1].

Lemma 4.4. *Let $p = (\emptyset, c_p, \Delta_p) \in \mathcal{P}_0$ and let $\alpha_0 \leq \alpha_1 < \beta < \omega_2$ be such that*

(1) *none of $\{\alpha_0, \beta\}$, $\{\alpha_1, \beta\}$ is in $\mathrm{dom}(c_p)$, and*
(2) *for every $N \in \mathrm{dom}(\Delta_p)$, $\{\alpha_0, \beta\} \in N$ if and only if $\{\alpha_1, \beta\} \in N$.*

Then there is $q = (\emptyset, c_q, \Delta_q) \leq_0 p$ such that $\{\alpha_0, \beta\}$, $\{\alpha_1, \beta\} \in \mathrm{dom}(c_q)$ and $c_q(\alpha_0, \beta) = c_q(\alpha_1, \beta)$.

Lemma 4.5. *$\mathcal{P}_0$ has the $\aleph_2$–c.c.*

Lemma 4.6. *Let $\theta > |\mathcal{P}_0|$ be a regular cardinal, $p = (\emptyset, c_p, \Delta_p) \in \mathcal{P}_0$, and N^* a countable elementary substructure of $H(\theta)$ such that p and Φ are in N^*. Let $N = N^* \cap H(\kappa)$ and let $p^+ = (\emptyset, c_p, \Delta_p \cup \{(N, 0)\})$. Then the following holds.*

(1) *$p^+ \leq_0 p$,*
(2) *If $q = (\emptyset, c_q, \Delta_q) \leq_0 p^+$ and $r = (\emptyset, c_q \restriction N, \Delta_q \cap N)$, then the following are true:*
   (2.1) *$r \in N \cap P_0$*
   (2.2) *If $s \in N \cap P_0$ and $s \leq_0 r$, then q and s are $\leq_0$–compatible.*
(3) *p^+ is an $(N^*, \mathcal{P}_0)$–generic condition.*

Lemma 4.6 implies of course that $\mathcal{P}_0$ is strongly proper.

The forcing $\mathcal{P}$ witnessing Theorem 4.2 is $(\mathcal{P}_\kappa, \leq_\kappa)$, where $\mathcal{P}_0$ is the poset in Definition 4.3 and $\langle(\mathcal{P}_\alpha, \leq_\alpha) \mid 1 \leq \alpha \leq \kappa\rangle$ is defined as follows.

Let α, $1 \leq \alpha \leq \kappa$, and suppose $\mathcal{P}_\xi$ has been defined for all $\xi < \alpha$. Conditions in $\mathcal{P}_\alpha$ are triples of the form $q = (F_q, c_q, \Delta_q)$ with the following properties.

(b0) F_q is a finite function with $\mathrm{dom}(F_q) \subseteq \alpha$.
(b1) Δ_q is a finite set of pairs of the form (N, γ), with N a countable elementary substructure of $(H(\kappa), \in, \Phi)$ and $\gamma \leq \alpha \cap \sup(N \cap \kappa)$.
(b2) For all $\xi < \alpha$, *the restriction of* q *to* ξ, denoted (as usual) by $q|_\xi$, is in $\mathcal{P}_\xi$. This restriction is defined as the triple

$$q|_\xi := (F_q \restriction \xi, c_q, \{(N, \min\{\gamma, \xi\}) : (N, \gamma) \in \Delta_q\})$$

(b3) If $\xi \in \mathrm{dom}(F_q)$, then $F_q(\xi) \in \mathbb{B}$.
(b4) If $\xi \in \mathrm{dom}(F_q)$, $(N, \beta) \in \Delta_q$, $\beta \geq \xi + 1$, and $N \in \mathcal{M}_{\xi+1}$, then δ_N is a fixed point of $F_q(\xi)$.

Given conditions $q^\epsilon = (F_\epsilon, c_\epsilon, \Delta_\epsilon)$ (for $\epsilon \in \{0, 1\}$) in $\mathcal{P}_\alpha$, $q^1 \leq_\alpha q^0$ if and only if

(c1) $q^1|_0 \leq_0 q^0|_0$,
(c2) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and, for all $\xi \in \mathrm{dom}(F_0)$, $F_1(\xi) \supseteq F_0(\xi)$, and
(c3) $\Delta_0 \subseteq \Delta_1$

The proofs of the corresponding versions of Lemmas 3.3, 3.4, 3.5 and 3.7 are identical to the proofs in [6]. In particular, the fact that the iterands in our present iteration are always forced to be $\mathbb{B}$, which is finitely proper, makes sure that the proof of the corresponding version of Lemma 3.7 goes through. One can also show that $\mathcal{P}_\kappa$ forces FA($\mathbb{B}$) by a standard argument using Lemmas 3.3 and 3.4 (cf. [6]).

It remains to prove that $\mathcal{P}_\kappa$ forces $\omega_2 \not\rightarrow_{\mathrm{poly}} (\omega_1)^2_{2\text{-bd}}$. This is an immediately consequence of Lemma 4.7. This lemma is a variant of a corresponding lemma in [1].

Lemma 4.7. *Let* $p \in \mathcal{P}_\kappa$ *and let* $\dot{X}$ *be a* $\mathcal{P}_\kappa$*–name. Assume that* p *forces that* $\dot{X}$ *is a subset of* ω_2 *of order type* ω_1*. Then there are* $t \leq_\kappa p$ *and ordinals* $\alpha_0 < \alpha_1 < \beta$ *in* $\mathrm{dom}(c_t)$ *such that*

(1) t *forces that* α_0, α_1 *and* β *are in* $\dot{X}$*, and*
(2) $c_t(\alpha_0, \beta) = c_t(\alpha_1, \beta)$.

Proof. By extending p if necessary, we may start assuming that there is some γ such that p forces that $\sup(\dot{X}) = \gamma$. Let θ be a sufficiently large regular cardinal and let $N^* \preccurlyeq H(\theta)$ be countable and such that p, $\mathcal{P}_\kappa$, Φ and $\dot{X}$ are in N^*. Let $N = N^* \cap H(\kappa)$ and let $p_0^+ \in \mathcal{P}_\kappa$ extend p

and such that $(N, \sup(N \cap \kappa)) \in \Delta_{p_0^+}$. By the relevant form of Lemma 3.7, p_0^+ is $(N^*, \mathcal{P}_\alpha)$–generic. Since $\mathrm{cf}(\gamma) = \omega_1$, $N \cap \gamma$ is bounded in γ. By extending p_0^+ if necessary we may assume that there is an ordinal β such that $\sup(N \cap \gamma) < \beta < \gamma$ and p_0^+ forces $\beta \in \dot{X}$. Let $m < \omega$ be bigger than the collection of all subsets of $\Delta_{p_0^+}$, let F_0 be the function with domain $\mathrm{dom}(F_{p_0^+}) \cap N$ and such that $F_0(\xi) = F_{p_0^+}(\xi) \restriction \delta_N$ for all $\xi \in \mathrm{dom}(F_0)$, and let $p_0 = (F_0, c_{p_0^+} \cap N, \Delta_{p_0^+} \cap N)$.

Now we can build increasing sequences of conditions $(p_i^+)_{1 \leq i \leq m}$ and $(p_i)_{1 \leq i \leq m}$, together with a sequence $(\alpha_i)_{1 \leq i \leq m}$ of distinct ordinals such that for all positive $i \leq m$,

(a) $p_i \in N$ extends p_{i-1} and forces $\alpha_i \in \dot{X}$,
(b) p_i^+ is a condition extending p_{i-1}^+,
(c) $\Delta_{p_i^+} \setminus N = \Delta_{p_{i-1}^+} \setminus N$, and
(d) $\mathrm{dom}(F_{p_i}) = \mathrm{dom}(F_{p_i^+}) \cap N$ and $F_{p_i}(\xi) = F_{p_i^+}(\xi) \restriction \delta_N$ for all $\xi \in \mathrm{dom}(F_{p_i})$, $c_{p_i} = c_{p_i^+} \cap N$, and $\Delta_{p_i} = \Delta_{p_i^+} \cap N$.

Given a positive $i \leq m$, if p_{i-1}^+ and p_{i-1} – namely, the restriction of p_{i-1}^+ to N – have been defined, we can find $p_i \in N$ by correctness of N since p_{i-1}^+ forces $\beta \in \dot{X}$ and $\beta \notin N$. But then we can easily find p_i^+. The point is that the forcings we are plugging in at nonzero stages ξ of the iteration are $\mathbb{B}$, so that, for every such ξ in $\mathrm{dom}(F_{p_i})$, it is indeed easy to find an extension of $F_{p_i}(\xi)$ which is generic for the relevant structures N' which are not in N simply by adding $\delta_{N'}$ as fixed point. The point is of course that we do not need to go to any extension of $p_i|_\xi$ to do that.

In the end we can find two i, $i' \leq m$ and an extension q of p_m^+ such that $c_q(\{\alpha_i, \beta\}) = c_q(\{\alpha_{i'}, \beta\})$ as in the proof (cf. [1]) of Lemma 4.4. More specifically, we may find distinct i, i' such that for all $M \in \mathrm{dom}(\Delta_{p_0^+})$, $\{\alpha_i, \beta\} \in M$ if and only if $\{\alpha_{i'}, \beta\} \in M$, and then we may fix some $\nu > e_\beta(\max\{\alpha_i, \alpha_{i'}\})$ such that $\nu \in M$ for all $M \in \mathrm{dom}(\Delta_{p_0^+})$ such that $\{\alpha_i, \beta\} \in M$ (equivalently, such that $\{\alpha_{i'}, \beta\} \in M$). Then we may let $c_q(\{\alpha_i, \beta\}) = c_q(\{\alpha_{i'}, \beta\}) = \nu$. This finishes the proof. $\square$

It is worth pointing out that if we were to force the forcing axiom for Cohen forcing rather than $\mathrm{FA}(\mathbb{B})$ (together with $2^{\aleph_0} = \kappa$), the construction could be made a bit simpler, in that we could dispense with the markers γ – in $(N, \gamma) \in \Delta_q$, for a condition q – in the definition of our iteration. Such a construction would of course suffice also to answer the question in [1].

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