RESEARCH STATEMENT

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The subject of my work is set theory, more precisely the study of forcing axioms and their effect on cardinal arithmetic. The origin of this subject is in the fundamental work of Kurt Godel and Paul Cohen who showed that Cantor's Continuum Problem (namely, the question: How many points are there on the real number line?) cannot be solved using only the usual Zermelo Fraenkel axioms (ZFC) of set theory. In the mathematical folklore, ZFC has been traditionally seen as the classical foundation of mathematics. This role has been given to this theory because all the mathematical assertions can be expressed in the language of ZFC and most of them can be decided using only the axioms of ZFC. Also, it may be argued that we believe in such axioms because they reflect our daily mathematical procedures and, in this sense, their use must not lead to any contradiction. Beside these external justifications, the axioms of ZFC can also be largely appreciated because they are the formalization of some ideas of Cantor's original work on transfinite (infinite) numbers, and such research is the historical motivation of set theory. So, this foundational aspect of set theory gives it philosophical as well as mathematical significance (even theological according to both, Cantor and some of his famous detractors). Anyway, and despite all its goodness, ZFC, as well as any other sufficiently powerful recursively axiomatized theory, is incomplete in the sense that there exist statements of the language of ZFC that can not be neither proved nor refuted using ZFC alone. Of course, incompleteness would not represent a serious problem if the undecidable assertions were not important from the mathematical point of view, but there exist some really interesting questions (like Cantor's Continuum Problem) which are in the origins of set theory and which have no answer in ZFC.

In order to show the independence of Cantor's Continuum Problem, Cohen invented in the 1960's the technique of forcing which is a very general method for producing models of set theory. Starting with a given universe V of ZFC and a partial ordering P in V one adjoins to V a P-generic filter G and obtains another universe of set theory V[G]. By carefully choosing the partial order P one can arrange that the Continuum Hypothesis (i.e., Cantor's conjecture that the size of the real line must be equal to the first uncountable number) or a number of other important statements are either true or false in V[G].

Given the profusion of independence results which followed Cohen's work it became a central objective of set theorists to find natural axiomatic extensions of ZFC which decide Cantor's problem as well as other important question undecidable in ZFC. For example, in the last five decades forcing axioms have been largely studied and have shown that they have very interesting consequences regarding the continuum (the continuum is defined as the infinite cardinal number that represents the size of the set of real numbers). Vaguely speaking these axioms assert that the universe of set theory is maximal in the sense that if a certain mathematical object can be adjointed by a suitable forcing notion, then it already exists. This is the analogous to the notion of an algebraically closed field. More precisely, given a class Γ of partial orders and a cardinal κ, the forcing axiom for Γ and κ, $\mathrm{FA}(\Gamma, \kappa)$, is the assertion that for every $\mathbb{P} \in \Gamma$ and every collection $\mathcal{D}$ of size at most κ consisting of maximal antichains of $\mathbb{P}$ (equivalently, for every collection $\mathcal{D}$ of size at most κ consisting of dense subsets of $\mathbb{P}$) there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Bounded forcing axioms – usually denoted by $\mathrm{BFA}(\Gamma)$, for a class Γ of partial orders – are the restricted forms of forcing axioms in which κ is equal to $\aleph_1$ and all antichains in $\mathcal{D}$ in the above formulation are taken to be of size at most $\aleph_1$. As it is well known, these statements are equivalent to principles of generic absoluteness for Σ_1 sentences over $H(\omega_2)$ with parameters.

Over the years a number of forcing axioms was proposed and studied by set theorists. Among them, in order of increasing strength are Martin's Axiom for $\aleph_1$ maximal antichains ($\mathrm{MA}_{\aleph_1}$) introduced by Solovay and Tennenbaum in the mid 1960's, the Proper Forcing Axiom (PFA) introduced by Baumgartner and Shelah in the early 1980's, the Semiproper Forcing Axiom (SPFA) and Martin's Maximum (MM) introduced by Foreman, Magidor and Shelah in the mid 1980's. They are defined as $\mathrm{FA}(\Gamma, \aleph_1)$ for Γ being, respectively, the class of all posets with the $\aleph_1$–chain condition ($\aleph_1$–c.c. for short), the class of all proper posets, the class of all semi-proper posets, and the class of all posets preserving stationary subsets of ω_1.[1] From the arithmetical perspective, the first application of forcing axioms is that, under $\mathrm{MA}_{\aleph_1}$, the continuum (i.e., $2^{\aleph_0}$) is equal to $2^{\aleph_1}$ which is of course strictly bigger

[1] SPFA is equivalent to MM.

than $\aleph_1$. However, neither $\mathrm{MA}_{\aleph_1}$ nor the full Martin's Axiom (MA) bound the continuum.[2]

One remarkable consequence of (bounded) forcing axioms is that, for reasonably natural classes Γ, $\mathrm{FA}(\Gamma, \aleph_1)$ ($\mathrm{BFA}(\Gamma)$) decides the size of the continuum, and in fact implies $2^{\aleph_0} = \aleph_2$. So far, the best result along these lines for bounded forcing axioms is Moore's theorem ([14]) that $\mathrm{BFA}(\{\mathbb{P} : \mathbb{P} \text{ proper}\})$ (also known as BPFA) implies $2^{\aleph_0} = \aleph_2$. In particular, PFA implies BPFA which in turn implies that the continuum is equal to the second uncountable cardinal. For unrestricted forcing axioms, the best result known is probably the older theorem, due to Todorčević and Veličković, that the forcing axiom for the class of posets of the form σ–closed $*$ $\aleph_1$–c.c. implies $2^{\aleph_0} = \aleph_2$. Interestingly, the corresponding implication for the bounded form of this theorem remains open. This is of course one of the questions that I would like to include here.

Question 0.1. *Let Γ be the class of all posets of the form σ–closed $*$ $\aleph_1$–c.c. Does $\mathrm{BFA}(\Gamma)$ imply $2^{\aleph_0} = \aleph_2$?*

In fact, it is not even known the following question.

Question 0.2. *Let Γ be the class of all ω–proper posets. Does $\mathrm{BFA}(\Gamma)$ imply $2^{\aleph_0} = \aleph_2$?*

For indecomposable ordinals $\alpha < \beta < \omega_1$, it can be shown that $\mathrm{BFA}(\Gamma_\alpha)$ does not follow from $\mathrm{FA}(\Gamma_\beta, \aleph_1)$, where Γ_α and Γ_β respectively denote the class of all the α–proper posets and the class of all the β–proper posets. This result and an old conjecture of Baumgartner refuted in [2][3] are the main motivation of the following question.

Question 0.3. *Let $\Gamma_{<\omega_1}$ be the class of all $< \omega_1$–proper[4] posets. Is $\mathrm{BFA}(\sigma\text{–closed} * \aleph_1\text{–c.c.})$ equivalent to $\mathrm{BFA}(\Gamma_{<\omega_1})$?*

There are another questions in the context of separation of forcing axioms. For instance:

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[2]Martin's axiom without parameters asserts that for every partial order P having the $\aleph_1$-chain condition, for every cardinal $\kappa < 2^{\aleph_0}$ and for every collection $\mathcal{D}$ of size at most κ consisting of maximal antichains of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Solovay and Tennenbaum proved that if ZFC is consistent, then so are the theories $\mathrm{ZFC}+\mathrm{MA}+2^{\aleph_0} = \aleph_{98}$, $\mathrm{ZFC}+\mathrm{MA}+2^{\aleph_0} = \aleph_{987}$, etc.

[3]After [2] has been done, we have learnt that in the 1980's Todorčević also refuted this conjecture with different methods from his results on the famous S–space problem; that proof has, however, never been published.

[4]i.e., α–proper for all indecomposable $\alpha < \omega_1$

Question 0.4. *Let us denote consider the restrictions of* MM *and* PFA *to partial orders of cardinality* $\aleph_1$. *Are they equivalent?* [5]

Question 0.5. *Is* BPFA *equivalent to* $\mathrm{FA}(\Gamma, \aleph_1)$*, where* Γ *is the class of all proper posets having the* $\aleph_2$*–chain condition?*

Let us focus now on the derivation of $2^{\aleph_0} = \aleph_2$ from "standard" forcing axioms. In all cases, this derivation involve some relevant form of generic absoluteness applied to forcing extensions via partial orders that collapse ω_2. A typical such argument goes along the following lines:

One considers a given real r and, by an application of the forcing axiom, finds an ordinal α in ω_2 which codes r in some specific sense. Since the definition of the coding is taken so that no ordinal can code two distinct reals, this shows $2^{\aleph_0} \leq \aleph_2$.[6] Given the particular nature of the coding, the existence of an ordinal α coding r can be expressed as a Σ_1 statement about r together with some fixed parameter $p \in H(\omega_2)$. One shows that there is a poset $\mathbb{P}$ in the class Γ forcing that there is such an ordinal κ. This ordinal is typically ω_2^V.[7] The poset collapses κ to be of size $\aleph_1$, and in fact the very definition of the coding entails that only ordinals less than ω_2 can code some real. The desired conclusion that there is an ordinal less than ω_2 coding r follows then from an application of the forcing axiom to $\mathbb{P}$.

In view of the above considerations it is natural to enquire whether the fact that the relevant poset $\mathbb{P}$ collapses cardinals above $\aleph_1$ is a necessary feature of every proof as above. Making the above enquiry more precise by restricting to the class of proper posets gives rise to the following question.

Question 0.6. *Does the restriction of* PFA *to cardinal–preserving proper posets imply* $2^{\aleph_0} = \aleph_2$*?*

Part of the progress in the study of forcing axioms includes the search for restricted forms of these axioms imposing limitations on the size of the set of the real numbers. Given that forcing axiom typically imply that the continuum is equal to the second uncountable cardinal, a natural problem when faced with a consequence C of a forcing axiom is

[5]Note that a partial order of cardinality ω_1 is proper iff it is semi–proper. The existence of a non–proper poset of size $\aleph_1$ preserving stationary subsets of ω_1 is consistent. In fact, Hiroshi Sakai constructed such a poset assuming a suitably strong version of $\diamondsuit_{\omega_1}$ holding in L and which can always be forced.

[6]$2^{\aleph_0} \geq \aleph_2$ follows trivially from the fact there is some poset in our class adding new reals.

[7]Or perhaps ω_4^V, or some fixed large cardinal in V. But always some ordinal at least ω_2^V.

to find out whether C itself has any impact on the size of the continuum, and which is this impact. This is of course a natural linearization problem. Note that if C does not bound this size, then there must exist models of C where the continuum is arbitrarily large. But the construction of such a model is a hard problem, since all the known techniques to force an interesting C[8] lead models where the continuum has size equal to $\aleph_2$. In [3] a breakthrough was achieved devising a new technique of finite support iteration which extends the known methods introduced by Solovay and Tennenbaum and which applies to a wide class of proper partial orders. This new approach consists in building, starting from a model satisfying the Continuum Hypothesis, a certain type of finite support forcing iteration of length $\kappa > \aleph_2$ (in a general sense of "forcing iteration") using what one may describe as finite "symmetric systems" of countable elementary substructures of a fixed $H(\kappa)$[9] as side conditions. These systems of structures are added at the first stage of the iteration. Roughly speaking, the fact that the supports of the conditions in the iteration are finite ensures that the inductive proofs of the relevant facts – mainly that the iteration has the $\aleph_2$–chain condition and that it is proper – go through. The use of the sets of structures as side conditions is crucial in the proof of properness. This is joint work with Asperó.

As a result, we gave in [3] a negative answer to a question of Moore [15] by showing that many consequences of BPFA are consistent together with the continuum being arbitrarily large. Other applications of our machinery (one of them solves a problem of Abraham and Cummings [1] in the context of polychromatic Ramsey theory) can be found in our recent paper [5]. So, I am optimistic about this approach also in the context of the following questions.

Question 0.7. *Does the consistency of* ZFC *imply the consistency of* MM *restricted to partial orders of cardinality* $\aleph_1$*?* [10]

Question 0.8. *Is it consistent the statement "All* $\aleph_2$*–dense sets of reals are isomorphic"?* [11]

[8] An *interesting* C typically follows from PFA and is strictly stronger than $MA_{\aleph_1}$.

[9] This κ is exactly the value that $2^{\aleph_0}$ attains at the end of the construction.

[10] The current consistency proof involves large cardinals, but very likely it can be improved.

[11] If μ is a cardinal, then a set A of real numbers is said to be μ–dense iff A intersects every interval in exactly μ points. Cantor showed that if A is $\aleph_0$–dense, then the structure $(A, <)$ (where $<$ denotes the usual order) is isomorphic to the set of the rational numbers. Also Baumgartner showed in [10] that, under PFA, all $\aleph_1$–dense sets are isomorphic.

An exciting line of research related with this second question is the search of higher analogues of the duo MA–PFA. As mentioned before, Martin's Axiom is forcing axiom for the class of all $\aleph_1$–c.c. posets and for less than $2^{\aleph_0}$ maximal antichains. Its first nontrivial[12] instance (i.e., MA relative to collections of at most $\aleph_1$ maximal antichains) implies that the continuum is at least $\aleph_2$, but it does not bound this value. However, PFA is a maximal form $\mathrm{MA}_{\aleph_1}$ implying that the continuum is equal to $\aleph_2$.

In [4] Asperó and I generalized Martin's Axiom to the class of posets satisfying what we call the $\aleph_{1.5}$–*chain condition* ($\aleph_{1.5}$–c.c. for short). This terminology is intended to highlight the fact that every poset with the $\aleph_1$–chain condition is in our class and that every poset in our class has the $\aleph_2$–chain condition. It follows from the first inclusion that for every cardinal λ, the forcing axiom $\mathrm{MA}^{1.5}_{\lambda} := \mathrm{FA}(\aleph_1\text{–c.c.}, \lambda)$ implies the forcing axiom $\mathrm{MA}_{\lambda} := \mathrm{FA}(\aleph_1\text{–c.c.}, \lambda)$. So, $\mathrm{MA}^{1.5}$ (defined as the conjunction of $\mathrm{MA}^{1.5}_{\lambda}$ for each λ below the continuum) is a generalization of MA. Furthermore, as to the consistency of $\mathrm{MA}^{1.5}_{\lambda}$, there is no restriction on λ other than $\lambda < 2^{\aleph_0}$. More precisely, the same construction shows that if ZFC is consistent, then so are the theories $\mathrm{ZFC} + \mathrm{MA}^{1.5} + 2^{\aleph_0} = \aleph_2$, $\mathrm{ZFC} + \mathrm{MA}^{1.5} + 2^{\aleph_0} = \aleph_{987}$, and so on.[13] This construction takes the form of a forcing iteration (again in a broad sense of the expression), but this time involving certain *partial* symmetric systems of countable submodels as side conditions (cf. our earlier work in [3]).

Note that the collapse of ω_1 to ω with finite conditions has size $\aleph_1$ and therefore has the $\aleph_2$–chain condition. The forcing axiom for collections of $\aleph_1$–many maximal antichains of this poset is obviously false. This shows of course that some restriction is necessary in order to obtain a consistent forcing axiom for posets with the $\aleph_2$–chain condition, even relative to collections of $\aleph_1$–many maximal antichains. On the other hand, the definition of our class of posets is wide enough to contain all $\aleph_1$–c.c. posets and in fact to make the corresponding forcing axiom $\mathrm{MA}^{1.5}_{\lambda}$ (for $\lambda \geq \aleph_1$) strictly stronger than MA_{λ}. In fact, $\mathrm{MA}^{1.5}_{\lambda}$ implies certain uniform λ–failures of Club Guessing on ω_1 that do not seem to have been considered before in the literature, and which do not follow from MA_{λ}. As a matter of fact we do not know how to show the

[12]In the sense that it does not follow from ZFC.

[13]The same is true for the Solovay–Tenennbaum construction, i.e., the same construction shows the consistency of Martin's Axiom together with $2^{\aleph_0}$ being $\aleph_2$, $\aleph_{987}$ and so on.

consistency of these λ–failures ($\lambda \geq \aleph_2$) by any method other than ours.

It is worth pointing out that Neeman [16] has developed a different method for building proper forcing notions by means of finite support iterations with side conditions. As a result, he proved (modulo large cardinals) the consistency the *relaxed two–size proper forcing axiom* (R2SPFA). Grosso modo this new axiom is a parallel of PFA but in the context of $\aleph_2$ maximal antichains. So, R2SPFA := FA$(\Gamma, \aleph_2)$, where Γ is the class of all relaxed two size proper posets. One similarity between PFA and R2SPFA is that these two principles respectively imply $\mathrm{MA}_{\aleph_1}$ and $\mathrm{MA}^{1.5}_{\aleph_2}$. Of course it is desirable to try to find a more suggestive analogy by showing that R2SPFA is a maximal form of $\mathrm{MA}^{1.5}_{\aleph_2}$ deciding the value of the continuum.

Question 0.9. *Does* R2SPFA *implies that the continuum is equal to* $\aleph_3$*?*

Question 0.10. *If the answer to the above question is affirmative, what are the foundational consequences of the existence of two incompatible forcing axioms, and in particular of two forcing axioms providing different values to the cardinality of the continuum?*

There is no doubt that in the last years there has been a second boom of the technique of forcing with side conditions (see for instance the recent works of Asperó–Mota, Krueger and Neeman describing three different perspectives of this technique). The first boom took place in the 1980's when Todorcevic [17] discovered a method of forcing in which elementary substructures are included in the conditions of a forcing poset to ensure that the forcing poset preserves cardinals. More than twenty years later, Friedman [11] and Mitchell [13] independently took the first step in generalizing the method from adding small (of size at most the first uncountable cardinal) generic objects to adding larger objects by defining forcing posets with finite conditions for adding a club subset on the second uncountable cardinal. However, neither of these results show how to force (with side conditions together with another finite set of objects) the existence of such a large object together with the continuum being small. In [11] Friedman asked whether it is possible to add a club subset of ω_2 with finite conditions while preserving the Continuum Hypothesis (CH). In [12] Krueger and I solved this problem by defining a forcing poset which adds a club to a fat stationary set and falls in the class of coherent adequate type forcings. Our main result is that any coherent adequate forcing preserves CH. Moreover, any coherent adequate forcing on $H(\lambda)$ (meaning that our side conditions are countable elementary substructures of $H(\lambda)$) , where $2^{\aleph_0} < \lambda$ is a cardinal of uncountable cofinality, collapses $2^{\aleph_0}$ to have size $\aleph_1$, preserves $(2^{\aleph_0})^+$, and forces CH. Therefore, it is natural to generalize Friedman's question in a higher context.

Question 0.11. *Is it possible to add a club subclass of the ordinals with finite conditions while preserving the Generalized Continuum Hypothesis?.*

Let me finish this research statement by describing my most recent advances concerning the possible behavior of CH within $H(\omega_2)$. This is joint work with Asperó. In [8], we introduced a new method for building models of CH, together with some particular Π_2 statements[14] over $H(\omega_2)$, by forcing. Unlike other iterated forcing constructions in the literature,[15] our construction adds new reals, although only $\aleph_1$-many of them. Using this approach, we build a model in which a principle isolated by Todorcevic known as Measuring holds together with CH, thereby answering a well-known question of Moore. This construction can be described as a finite-support weak forcing iteration with side conditions consisting of suitable graphs of sets of models with markers. The CH-preservation is accomplished through the imposition of copying constraints on the information carried by the condition, as dictated by the edges in the graph. A minor variation of the main construction produces a forcing notion giving rise to a model of Measuring together with $2^{\aleph_0}$ being arbitrarily large. This answers another question of Moore.

This latest work has a long and somewhat turbulent history which I would like to summarize in the following lines. I start out by mentioning [6], published in the Journal of Symbolic Logic in 2017. In this paper we were allegedly constructing, by means of a forcing with symmetric systems of models with markers as side conditions, a model of Measuring together with $2^{\aleph_0}$ being arbitrarily large. That same year, a colleague of us found out that our main proof in that paper had an error. His argument did not exhibit an actual counterexample to the relevant claims in our proof in [6], but it did show, at the very least, that our proof in [6] was incomplete as it did not establish those claims. This eventually prompted us to retract [6] ([7]).

In a parallel project, Asperó and I were working on forcing Measuring together with CH, also using side conditions. The project culminated

[14]Note that these statements are of the form $\forall x \exists y \alpha(x, y)$ and therefore, they typically imply a large number of (existential) realizations to be true.

[15][12] and [8] were both inspired by a symmetric argument which appeared first in [5], but the construction in [12] is by no means an iteration since it involves only one poset.

in a theorem stating that the consistency of ZFC[16] implies the consistency of ZFC + Measuring + CH, which we view as the main theorem of [8]. The connection with the first story is the following: in the summer of 2022, we realized that a very mild variant of our forcing witnessing the above theorem actually gives rise to a model of Measuring together with $2^{\aleph_0} > \aleph_2$; in fact, by removing one of the clauses in the definition of the forcing for the main theorem of [8] we produce a cardinal-preserving forcing giving rise to a model of Measuring together with $2^{\aleph_0} = \kappa$, where κ is an arbitrarily chosen regular cardinal. We decided then to include this second construction in the same paper, which salvages the main result from [6]. Overall, my impression is that our new technology is flexible enough to be adapted to some other principles giving rise to new consistency results in both environments, with CH and with a large continuum.

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