A new computational method for locating the secondary bifurcation of a Rankine-Hugoniot curve

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Terse review of Riemann Problems
The Rankine-Hugoniot locus

Consider a 2×2 conservation laws system

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u,\, v)}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial g(u,\, v)}{\partial x} &= 0, \end{aligned}$$

for $u,\, v : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$. Let $U := (u,\, v)^{\mathrm{T}}$ and $F := (f,\, g)^{\mathrm{T}}$.

The mass balance for a shock with speed $\sigma = \sigma(U^o,\, U)$ is

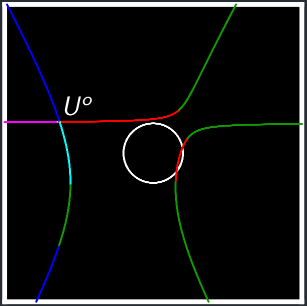
$$\begin{aligned} f(U) - f(U^o) &= \sigma(u - u^o), \\ g(U) - g(U^o) &= \sigma(v - v^o), \end{aligned}$$

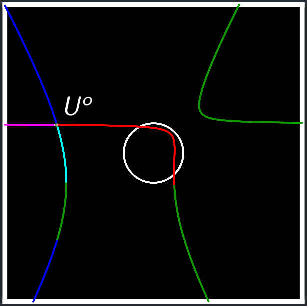
the Rankine-Hugoniot (RH) relation $[F] = \sigma[U]$.

For fixed U^o, U parametrizes the Hugoniot locus $\mathcal{H}(U)$.

The "Hugoniot" plot...
... the case of the first bifurcation case

Typically Hugoniot loci are not curves: self-intersection at U^o.





What happens in between?

The "Hugoniot" plot...
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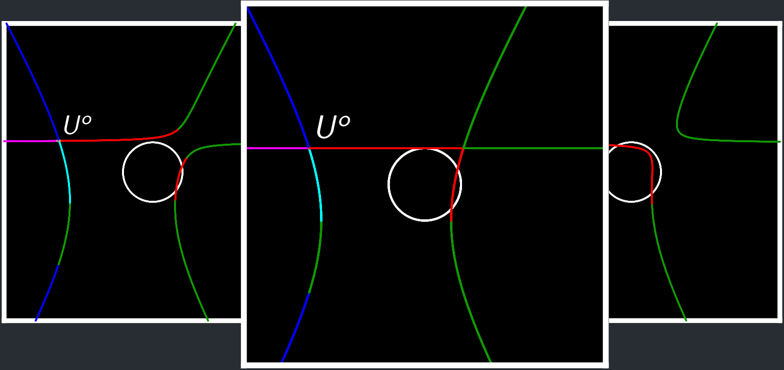


Another bifurcation!

Terse review of Riemann Problems
The "Rankine-Hugoniot function"

Cancelling speed σ from RH relations $[F] = \sigma[U]$, leads to

$$[v - v^o][f(U) - f(U^o)] - [u - u^o][g(U) - g(U^o)] = 0,$$

the zero root of which is $\mathcal{H}(U^o)$.

We define the Rankine-Hugoniot function as

$$RH(U; U^o) := [v - v^o][f(U) - f^o] - [u - u^o][g(U) - g^o].$$

Terse review of Riemann Problems
The Secondary Bifurcation manifold

New bifurcations rely on intrinsic relations...

The Jacobian matrix:	$DF(U)$	$:=$	$\frac{\partial F}{\partial U}$	$=$	$\frac{\partial(f,g)^{\mathrm{T}}}{\partial(u,v)^{\mathrm{T}}}$
Eigenvalue order by "family":			$\lambda_1(U)$	$\leq$	$\lambda_2(U)$
Associated (rigth) eigenvectors:			$r_1(U)$		$r_2(U)$
Associated (left) eigenvectors:			$\ell_1(U)$		$\ell_2(U)$

Definition (Shearer and Schaeffer (1987))

A state U belongs to the **secondary bifurcation manifold** if there exists a state $U' \neq U$ such that

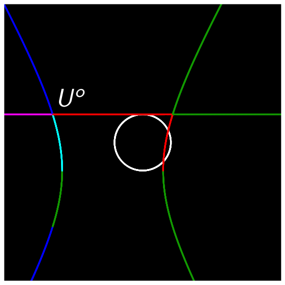
$$U' \in \mathcal{H}(U) \ \text{ with } \ \lambda_i(U') = \sigma(U,\, U') \ \text{ and } \ \ell_i(U')(U' - U) = 0.$$

(The manifolds may be related to families.)

Terse review of Riemann Problems
The Secondary Bifurcation manifold

We want all sates U with corresponding U'...
a curve in $\mathbb{R}^4$.

Numerical method may solve three relations.



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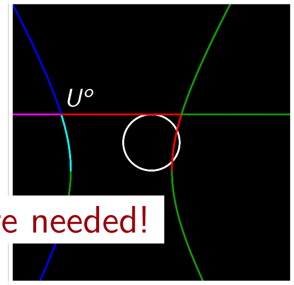

Eigenvalues and eigenvectors are needed!

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(The manifolds may be related to families.)

Introduction to Algebraic Geometry
Plane curves

The RH function is an affine plane curve; $RH \in k[\mathbb{R}, \mathbb{R}]$.

Definition (W. Fulton (1989))

The point U is called *simple* of RH if either derivative $RH_u(U) \neq 0$ or $RH_v(U) \neq 0$. A point that it is not simple is called *singular*.

Actually the "tangent line" to RH at U is

$$RH_u(U)(U - x) + RH_v(U)(v - y) = 0.$$

Numerical method may solve three algebraic relations!

Introduction to Algebraic Geometry

Locus by three little equations...

The numerical method looks for a zero root of algebraic equations:

$$\begin{aligned}
\text{RH function:} \quad RH(U;\, U^o) &= [v][f]-[u][g],\\
\partial RH/\partial u: \quad RH_u(U;\, U^o) &= [v]f_u(U)-[u]g_u(U)-[g],\\
\partial RH/\partial v: \quad RH_v(U;\, U^o) &= [v]f_v(U)-[u]g_v(U)+[f],
\end{aligned}$$

where $[u]=u-u^o$, $[v]=v-v^o$, $[g]=g(U)-g^o$, $[f]=f(U)-f^o$.

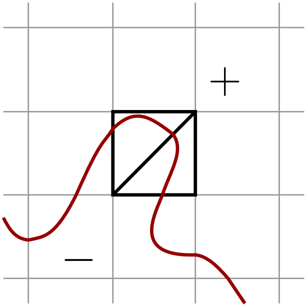
Instead of difficult expressions:

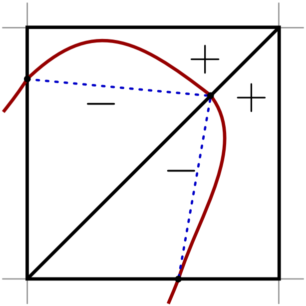
$$\begin{aligned}
\text{RH function:} \quad [v][f]-[u][g] &= 0,\\
\text{speed matching:} \quad \lambda_i(U)-\sigma(U^o,\, U) &= 0,\\
\text{vector orthogonality:} \quad \ell_i(U)(U-U^o) &= 0.
\end{aligned}$$

However, we lost family association.

Piecewise linear approximation of the RH relation
Contour 2D

Numerical grid splits the domain into cells; triangular configuration.





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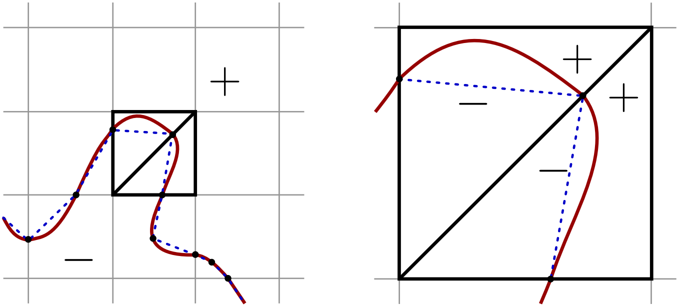
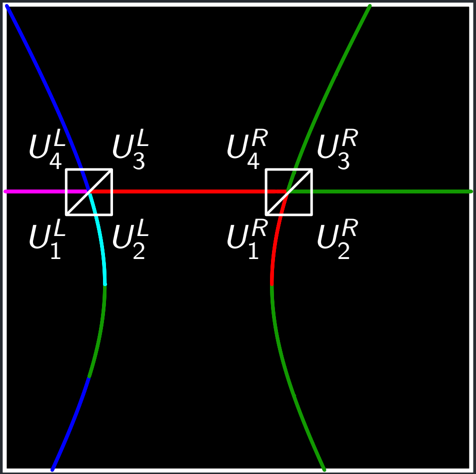


For elements in $\mathbb{R}^4$ we will use simplex.

Simplex configuration
Contour 2×2

Pairs U^L, U^R conform the secondary manifold.



Simplex configuration
Contour 2×2

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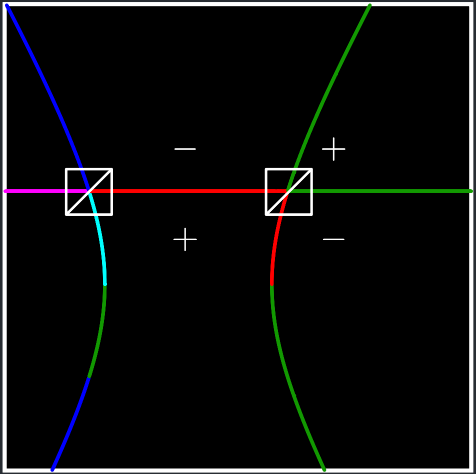
RH function

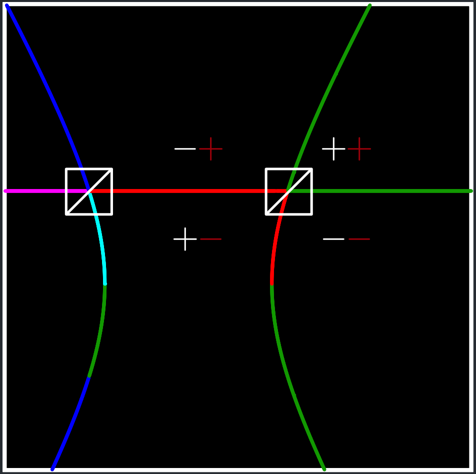


Simplex configuration
Contour 2×2

Pairs U^L, U^R conform the secondary manifold.

RH function

$\frac{\partial RH}{\partial u}$



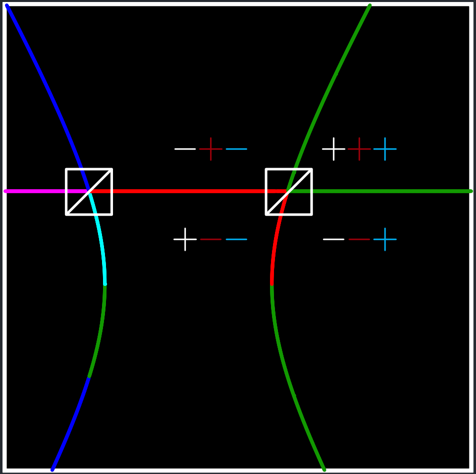
Simplex configuration
Contour 2×2

Pairs U^L, U^R conform the secondary manifold.

RH function

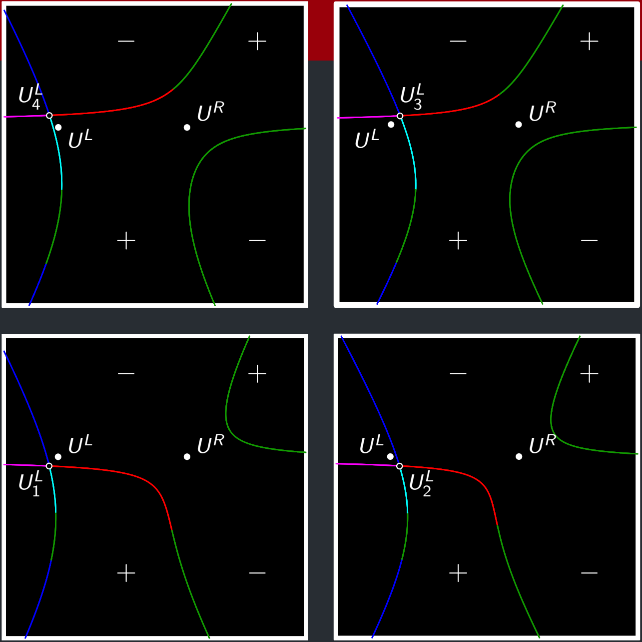
$\frac{\partial RH}{\partial u}$

$\frac{\partial RH}{\partial v}$

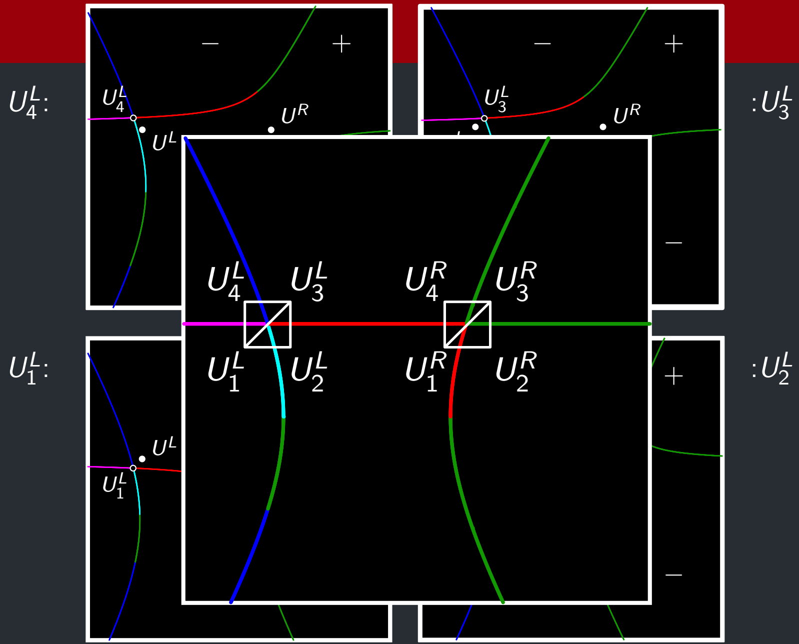


U_4^L: $:U_3^L$

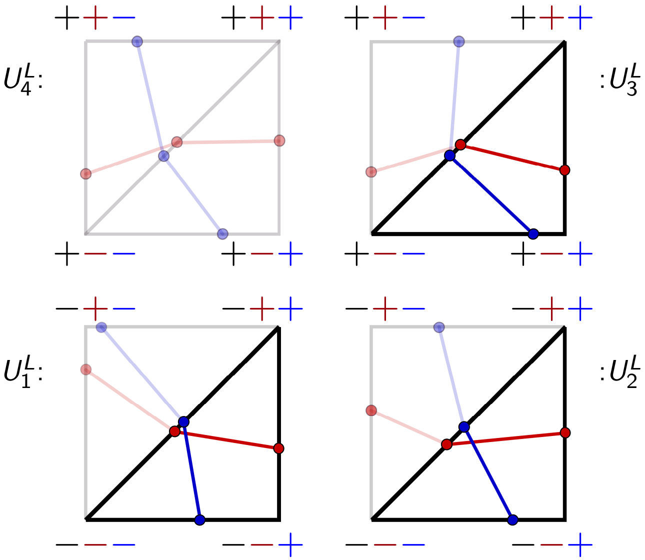
U_1^L: $:U_2^L$



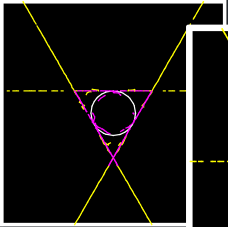
Numerical approach



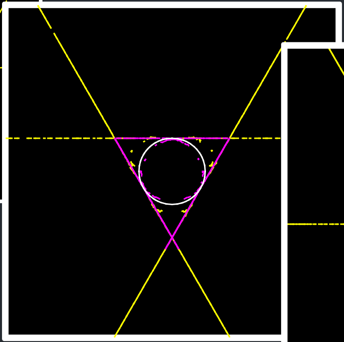
Simplex configuration



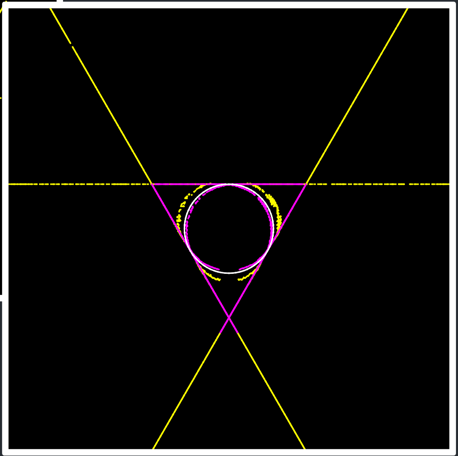
The quadratic model
Elliptic region presence



30 × 30



60 × 60

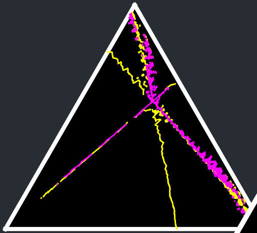


120 × 120

$$f(u,v) = \frac{1}{2}\left(-u^2 + v^2 + 0.1v\right)$$

$$g(u,v) = \frac{1}{2}\left(2uv - 0.1u\right)$$

Corey quadratic model



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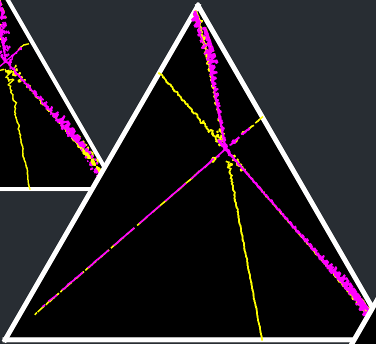

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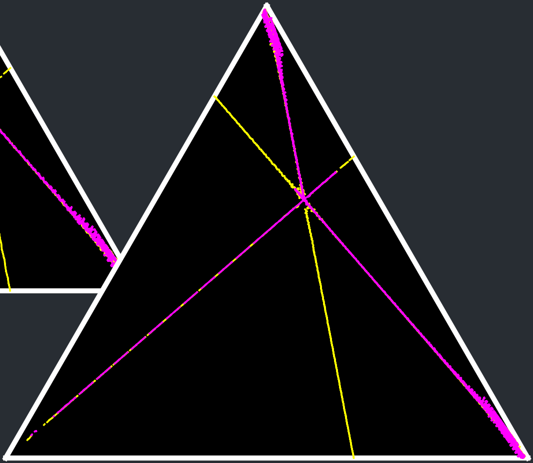

120 × 120

$$f(u,v) = \frac{u^2}{u^2+2v^2+(1-u-v)^2/2}$$

$$g(u,v) = \frac{2v^2}{u^2+2v^2+(1-u-v)^2/2}$$

$\mathcal{T}$hank you for your attention!