Solution to the Closed Economy Heckscher-Ohlin Model

ECO-13101 Economia Internacional I (International Trade Theory)*

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Two Good Two Country HO Model of Trade: Closed Economy or Autarky

Consider a world with two countries, home (h) and foreign (f). Both countries produce two goods, X and Y, using two inputs, labor (L) and capital (K). Output of each good in country $j = \{h, f\}$ is given by the following production function:

$$X_p^j = \left(K_x^j\right)^{\alpha}\left(L_x^j\right)^{1-\alpha} \quad ,$$

$$Y_p^j = \left(K_y^j\right)^{\beta}\left(L_y^j\right)^{1-\beta} \quad ,$$

where the subscript p denotes production of goods, and $0 < \alpha, \beta < 1$. α and β represents the share of capital in the total income (value added) in sector X and sector Y, respectively, in each country. The total endowment of labor in home is $\overline{L}^h = 1$ and that in foreign is $\overline{L}^f = 2$. Similarly, the endowments of capital are $\overline{K}^h = 3$ and $\overline{K}^f = 2$. Furthermore, the two countries have identical and homogeneous preferences, which are given by:

$$U(X, Y) = X_c^{\delta} Y_c^{1-\delta} \quad ,$$

where the subscript c denotes consumption of goods. $\delta = 0.5$, and it represent the share of good X in total expenditure in each country. Suppose $\beta = 0.5$ and $\alpha = 0.3$.

The solution to the model involves three components. First, given the goods' prices and factor prices, consumer maximizes utility subject to the budget constraint. Second, given the goods' prices and factor prices, firms in the two sectors maximize profits. Lastly, the markets clear - (i) the goods' prices are such that the demand for each good by the consumers must equal its supply by the firm, and (ii) the factor prices are such that the demand for each factor by the firm must equal its supply by the consumer.

Utility maximization by the consumer in country j:

$$\max_{\{X_c^j, Y_c^j\}} \left(X_c^j\right)^{\delta}\left(Y_c^j\right)^{(1-\delta)}$$

$$s.t. \;\; p_x^j X_c^j + p_y^j Y_c^j = w_j \overline{L}^j + r_j \overline{K}^j \quad (Budget\ Constraint).$$

Upon setting up the lagrangian, we get the following first order conditions with respect to X_c^j and Y_c^j, respectively:

$$\delta\left(X_c^j\right)^{(\delta-1)}\left(Y_c^j\right)^{(1-\delta)} = \lambda^j p_x^j \quad ,$$

$$(1-\delta)\left(X_c^j\right)^\delta\left(Y_c^j\right)^{-\delta} = \lambda^j p_y^j \quad .$$

Dividing one by the other helps us to get rid of the lagrange multiplier, λ, and gives the ratio of the expenditures on each good.

$$\frac{p_x^j X_c^j}{p_y^j Y_c^j} = \frac{\delta}{(1-\delta)} \quad . \tag{1}$$

The first order condition with respect to the lagrange multiplier gives nothing but the budget constraint.

$$p_x^j X_c^j + p_y^j Y_c^j = w_j \overline{L}^j + r_j \overline{K}^j \tag{2}$$

Firm's profit maximization in sector X in country j.

$$\max_{\{L_x^j, K_x^j\}} p_x^j \left(K_x^j\right)^\alpha \left(L_x^j\right)^{1-\alpha} - w_j L_x^j - r_j K_x^j \quad .$$

The first order conditions with respect to labor and capital, respectively, are:

$$p_x^j(1-\alpha)\left(K_x^j\right)^\alpha\left(L_x^j\right)^{-\alpha} = w_j \quad , \tag{3}$$

$$p_x^j \alpha \left(K_x^j\right)^{\alpha-1}\left(L_x^j\right)^{1-\alpha} = r_j \quad . \tag{4}$$

Firm's profit maximization in sector Y in country j.

$$\max_{\{L_y^j, K_y^j\}} p_y^j \left(K_y^j\right)^\beta \left(L_y^j\right)^{1-\beta} - w_j L_y^j - r_j K_y^j \quad .$$

The first order conditions with respect to labor and capital, respectively, are:

$$p_y^j(1-\beta)\left(K_y^j\right)^\beta\left(L_y^j\right)^{-\beta} = w_j \quad , \tag{5}$$

$$p_y^j \beta \left(K_y^j\right)^{\beta-1}\left(L_y^j\right)^{1-\beta} = r_j \quad . \tag{6}$$

Market clearing condition for goods in country j are given by

$$X_c^j = X_p^j = X^j \quad , \tag{7}$$

$$Y_c^j = Y_p^j = Y^j \quad , \tag{8}$$

and **that for labor and capital** are given by

$$L_x^j + L_y^j = \overline{L}^j \quad , \tag{9}$$

$$K_x^j + K_y^j = \overline{K}^j \quad . \tag{10}$$

Eq. (1) - Eq. (10) forms a system of 10 equations needed to solve for 10 unknowns for each country. However, by Walras' law one of the goods market clearing conditions is redundant. As a result, we are short of one equation. So, we normalize $w_j = 1$ for $j \in \{h, f\}$.

1. Substituting Eq. (1) in the budget constraint gives the share of total income that the consumer in country j spends on each good. Note that this nice result is driven by the Cobb-Douglas utility function we have started off with.

   $$p_x^j X_c^j = \delta \left[w_j \overline{L}^j + r_j \overline{K}^j \right] \quad , \tag{11}$$

   $$p_y^j Y_c^j = (1 - \delta) \left[w_j \overline{L}^j + r_j \overline{K}^j \right] \quad . \tag{12}$$

   The term in the square bracket on the right-hand side (in both equations) is the total income of the consumer.

   Multiplying both sides of Eq. (3) by L_x^j and substituting $X_p^j = (K_x^j)^\alpha (L_x^j)^{(1-\alpha)}$ gives the share of labor in the total income of sector X.

   $$\frac{w_j L_x^j}{p_x^j X_p^j} = 1 - \alpha \quad . \tag{13}$$

   Similarly, the share of capital in total income of sector X is:

   $$\frac{r_j K_x^j}{p_x^j X_p^j} = \alpha \quad . \tag{14}$$

   Carrying out the same exercise for labor and capital in sector Y gives:

   $$\frac{w_j L_y^j}{p_y^j Y_p^j} = 1 - \beta \quad , \tag{15}$$

   $$\frac{r_j K_y^j}{p_y^j Y_p^j} = \beta \quad . \tag{16}$$

   Imposing the goods' market clearing conditions and dividing Eq. (13) by Eq. (15) gives:

   $$\frac{L_x^j}{L_y^j} . \frac{p_y^j Y^j}{p_x^j X^j} = \frac{1 - \alpha}{1 - \beta} \quad .$$

Using Eq. (1) and rearranging gives the relationship between allocation of labor across the two sectors.

$$L_x^j = \frac{\delta}{(1-\delta)}.\frac{(1-\alpha)}{(1-\beta)}.L_y^j \quad .$$

Substituting for L_x^j in the labor market clearing condition then gives us allocation of the labor across sector X and Y.

$$L_x^j = \frac{\delta(1-\alpha)}{(1-\delta)(1-\beta)+\delta(1-\alpha)}.\overline{L}^j \quad , \tag{17}$$

$$L_y^j = \frac{(1-\delta)(1-\beta)}{(1-\delta)(1-\beta)+\delta(1-\alpha)}.\overline{L}^j \quad . \tag{18}$$

Following the same procedure for capital gives us the allocation of capital across the two sectors.

$$K_x^j = \frac{\alpha\delta}{\beta(1-\delta)+\alpha\delta}.\overline{K}^j \quad , \tag{19}$$

$$K_y^j = \frac{\beta(1-\delta)}{\beta(1-\delta)+\alpha\delta}.\overline{K}^j \quad , \tag{20}$$

Divide Eq. (3) by Eq. (4) (and Eq. (5) by Eq. (6)) to get the wage-rental ratio as a function of the capital-labor ratio of each sector.

$$\frac{w_j}{r_j} = \frac{(1-\alpha)}{\alpha}.\frac{K_x^j}{L_x^j} = \frac{(1-\beta)}{\beta}.\frac{K_y^j}{L_y^j} \quad . \tag{21}$$

Use Eq. (17) and Eq. (19) to substitute for L_x^j and K_x^j and remember that $w_j = 1$. This gives us the rental-rate on capital as function of country j's capital-labor ratio.

$$r_j = \frac{\alpha\delta+\beta(1-\delta)}{1-(\alpha\delta+\beta(1-\delta))}.\frac{\overline{L}^j}{\overline{K}^j} \tag{22}$$

To get the autarky price ratio, divide Eq. (3) by Eq. (5) to get

$$p_j^a = \frac{p_x^j}{p_y^j} = \frac{(1-\beta)}{(1-\alpha)}.\frac{\left(K_y^j/L_y^j\right)^\beta}{\left(K_x^j/L_x^j\right)^\alpha} \quad .$$

Using Eq. (21) to substitute for capital-labor ratios of the two sectors with the wage-rental ratio implies that the price ratio is given by:

$$p_j^a = \frac{p_x^j}{p_y^j} = \frac{\beta^\beta}{\alpha^\alpha}.\frac{(1-\beta)^{(1-\beta)}}{(1-\alpha)^{(1-\alpha)}}.r_j^{(\alpha-\beta)} \quad ,$$

which after substituting for r_j from Eq. (22) gives:

$$p_j^a = \frac{p_x^j}{p_y^j} = \frac{\beta^\beta}{\alpha^\alpha} \cdot \frac{(1-\beta)^{(1-\beta)}}{(1-\alpha)^{(1-\alpha)}} \cdot \left[\frac{1-(\alpha\delta+\beta(1-\delta))}{\alpha\delta+\beta(1-\delta)}\right]^{(\beta-\alpha)} \cdot \left[\frac{\overline{K}^j}{\overline{L}^j}\right]^{(\beta-\alpha)} . \quad (23)$$

So, the autarky price ratio of country j is directly proportional to the capital-labor ratio of the two countries (since $\beta > \alpha$), besides being a function of some parameters that are constant and are common to the two countries. Therefore, the capital abundant (relative to labor), country, i.e. country with higher capital-labor ratio, will have a higher relative price of good X (the labor intensive good). Since $\overline{K}^h/\overline{L}^h = 3$ and $\overline{K}^f/\overline{L}^f = 1$, this implies that the labor intensive good X is more expensive relative to the capital intensive good Y in the capital-abundant country H.

2. To compute the utility levels, we need to find the absolute price levels of good X and good Y in each country j. Using the first order conditions from the firm's maximization, Eq. (3), and substituting for capital-labor ratios from Eq. (21) gives

$$p_x^j = \frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}} . r_j^\alpha \quad ,$$

which after substituting for r_j from Eq. (22) becomes

$$p_x^j = \frac{1}{\alpha^\alpha(1-\alpha)^{(1-\alpha)}} \cdot \left[\frac{\alpha\delta+\beta(1-\delta)}{1-(\alpha\delta+\beta(1-\delta))}\right]^\alpha \cdot \left[\frac{\overline{L}^j}{\overline{K}^j}\right]^\alpha .$$

Similarly the autarky price of good Y is given by:

$$p_y^j = \frac{1}{\beta^\beta(1-\beta)^{(1-\beta)}} \cdot \left[\frac{\alpha\delta+\beta(1-\delta)}{1-(\alpha\delta+\beta(1-\delta))}\right]^\beta \cdot \left[\frac{\overline{L}^j}{\overline{K}^j}\right]^\beta .$$

Use these expressions for price levels and for rental rate (Eq. (22)) in the equations from consumer's utility maximization exercise that give the share of the expenditure on good X and good Y in total income - Eq. (11) and Eq. (12) - to get the consumption of good X and Y.

$$X^j = \delta A \left(\overline{K}^j\right)^\alpha \left(\overline{L}^j\right)^{(1-\alpha)} \quad ,$$

$$Y^j = (1-\delta) B \left(\overline{K}^j\right)^\beta \left(\overline{L}^j\right)^{(1-\beta)} \quad ,$$

where

$$A = \frac{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}{\left[\alpha\delta+\beta(1-\delta)\right]^{\alpha}\left[1-(\alpha\delta+\beta(1-\delta))\right]^{(1-\alpha)}} \quad , \quad and$$

$$B = \frac{\beta^{\beta}(1-\beta)^{(1-\beta)}}{\left[\alpha\delta+\beta(1-\delta)\right]^{\beta}\left[1-(\alpha\delta+\beta(1-\delta))\right]^{(1-\beta)}} \quad .$$

Substituting these expressions in the expression for the utility function gives the autarky utility level for country j.

$$U_j^a = \delta^{\delta}(1-\delta)^{(1-\delta)}A^{\delta}B^{(1-\delta)}\left(\overline{K}^j\right)^{(\alpha\delta+\beta(1-\delta))}\left(\overline{L}^j\right)^{((1-\alpha)\delta+(1-\beta)(1-\delta))} \quad . \tag{24}$$