Question 2:

(2) Consumer in country i

$\therefore$ Max $U_i = \left[T_{ii}^{\rho} + T_{ij}^{\rho} \right]^{\theta/\rho} . N_i^{1-\theta}$

F.O.Cs

T_{ii}: $\frac{\theta}{\rho} \left[T_{ii}^{\rho} + T_{ij}^{\rho} \right]^{\frac{\theta}{\rho}-1} . \rho \, T_{ii}^{\rho-1} . N_i^{1-\theta} = \lambda_i . p_{ii}^{T}$

T_{ij}: $\frac{\theta}{\rho} \left[T_{ii} + T_{ij}^{\rho} \right]^{\frac{\theta}{\rho}-1} . \rho \, T_{ij}^{\rho-1} . N_i^{1-\theta} = \lambda_i \, p_{ij}^{T}$

N_i: $(1-\theta) \left[T_{ii}^{\rho} + T_{ij}^{\rho} \right]^{\theta/\rho} . N_i^{-\theta} = \lambda_i \, p_i^{N}$

Then using the first two

$$\left(\frac{T_{ij}}{T_{ii}}\right)^{\rho-1} = \frac{p_{ij}^{T}}{p_{ii}^{T}} = \frac{(1+\tau_{ij})\, p_j^{T}}{p_i^{T}}$$

$$\Rightarrow \left(\frac{(1+\tau_{ij}) p_j^{T} T_{ij}}{p_i^{T} T_{ii}}\right)^{\rho-1} . \frac{(p_i^{T})^{\rho-1}}{((1+\tau_{ij}) p_j^{T})^{\rho-1}} = \frac{(1+\tau_{ij})\, p_j^{T}}{p_i^{T}}$$

Let $X_{ij} = (1+\tau_{ij})\, p_j^{T}\, T_{ij}$

$X_{ii} = p_i^{T} . T_{ii}$

$$\therefore \left(\frac{X_{ij}}{X_{ii}}\right)^{\rho-1} = \frac{(1+\tau_{ij})^{\rho} . (p_j^{T})^{\rho}}{(p_i^{T})^{\rho}}$$

$$\Rightarrow \left(\frac{X_{ii}}{X_{ij}}\right)^{1-\rho} = \frac{(1+\tau_{ij})^{\rho} . (p_j^{T})^{\rho}}{(p_i^{T})^{\rho}} = (1+\tau_{ij})^{\rho} . \frac{(w_j/\alpha_j)^{\rho}}{(w_i/\alpha_i)^{\rho}}$$

Interpret this equation $\Rightarrow$ $$\boxed{\log\left(\frac{X_{ii}}{X_{ij}}\right) = \frac{\rho}{1-\rho} . \log\left(\frac{w_j}{w_i}\right) + \frac{\rho}{1-\rho} . \log\left(\frac{\alpha_i}{\alpha_j}\right) + \frac{\rho}{1-\rho} . \log(1+\tau_{ij})}$$

What is the elasticity of ~~tra~~ relative trade volume w.r.t $(1+\tau_{ij})$

$$\frac{\partial \log(X_{ii}/X_{ij})}{\partial \log(1+\tau_{ij})} = \frac{\rho}{1-\rho}$$

From the consumer's F.O.Cs

$$\frac{\theta}{1-\theta}\cdot\frac{N_i}{[T_{ii}^{\rho}+T_{ij}^{\rho}]\,T_{ii}^{1-\rho}} = \frac{p_i^T}{p_i^N}$$

$$\Rightarrow \frac{\theta}{1-\theta}\cdot\frac{N_i}{T_{ii}}\cdot\frac{1}{\left[1+\left(\frac{T_{ij}}{T_{ii}}\right)^{\rho}\right]} = \frac{p_i^T}{p_i^N}$$

Similarly

$$\frac{\theta}{1-\theta}\cdot\frac{N_i}{T_{ij}}\cdot\frac{1}{\left[\left(\frac{T_{ii}}{T_{ij}}\right)^{\rho}+1\right]} = \frac{(1+\tau_{ij})p_j^T}{p_i^N}$$

Adding the two equations implies

$$\frac{\theta}{1-\theta}N_i\left[\frac{1}{T_{ii}^{1-\rho}[T_{ii}^{\rho}+T_{ij}^{\rho}]}+\frac{1}{T_{ij}^{1-\rho}[T_{ii}^{\rho}+T_{ij}^{\rho}]}\right] = \frac{1}{p_i^N}\left[p_i^T+(1+\tau_{ij})p_j^T\right]$$

$$\frac{\theta}{1-\theta}N_i$$

(3) Note that $p_i^N \cdot N_i = (1-\theta)w_i$

By market clearing $N_i = Y_i^N$

$$\Rightarrow p_i^N Y_i^N = (1-\theta)w_i$$

Since firms use a CRS technology $\Rightarrow$ profits are zero

$$\Rightarrow \cancel{w_i}\cdot L_i^N = (1-\theta)\cancel{w_i}$$

$$\therefore L_i^N = (1-\theta)$$

Since $L_i^T + L_i^N = 1$

$$\Rightarrow L_i^T = \theta$$

Note that

$$p_i^T \cdot T_{ii} + p_j^T(1+\tau_{ij})T_{ij} = \theta\cdot w_i$$

Since $X_{ii} = \left[(1+\tau_{ij})^{\rho}\left(\frac{p_j^T}{p_i^T}\right)^{\rho}\right]^{1/1-\rho}\cdot X_{ij}$

$$\Rightarrow \left[(1+\tau_{ij})\frac{p_j^T}{p_i^T}\right]^{\rho/(1-\rho)} X_{ij} + X_{ij} = \theta w_i$$

$$\Rightarrow \quad X_{ij} = \frac{\theta w_i}{\left[(1+\tau_{ij})\frac{p_j^T}{p_i^T}\right]^{\rho/(1-\rho)}+1}$$

Similarly $X_{ji} = \dfrac{\theta w_j}{\left[(1+\tau_{ji})\cdot\frac{p_i^T}{p_j^T}\right]^{\rho/(1-\rho)}+1}$

Since trade is balanced $\quad X_{ij} = X_{ji}$

$$\frac{\theta w_i}{\left[(1+\tau_{ij})\cdot\frac{p_j^T}{p_i^T}\right]^{\rho/(1-\rho)}+1} = \frac{\theta w_j}{\left[(1+\tau_{ji})\cdot\frac{p_i^T}{p_j^T}\right]^{\rho/(1-\rho)}+1}$$

Let $i=h$ & $j=f$, and assume $w_h = 1$

$$\Rightarrow \left[(1+\tau_{ji})\frac{p_i^T}{p_j^T}\right]^{\rho/(1-\rho)}+1 = w_j\left\{\left[(1+\tau_{ij})\cdot\frac{p_j^T}{p_i^T}\right]^{\rho/(1-\rho)}+1\right\}$$

$$\Rightarrow (1+\tau_{fh})^{\rho/(1-\rho)}\cdot\frac{(w_h/\alpha_h)^{\rho/(1-\rho)}}{(w_f/\alpha_f)^{\rho/(1-\rho)}}+1 = w_f\,(1+\tau_{hf})^{\rho/(1-\rho)}\cdot\frac{(w_f/\alpha_f)^{\rho/(1-\rho)}}{(w_h/\alpha_h)^{\rho/(1-\rho)}}+w_f$$

$$\Rightarrow (1+\tau_{fh})^{\rho/(1-\rho)}\cdot\left(\frac{\alpha_f}{\alpha_h}\right)^{\rho/(1-\rho)}\cdot w_f^{\rho/(\rho-1)}+1 = (1+\tau_{hf})^{\rho/(1-\rho)}\left(\frac{\alpha_f}{\alpha_h}\right)^{\rho/(\rho-1)}\cdot w_f^{1/(1-\rho)}+w_f$$

$$\Rightarrow \boxed{(1+\tau_{fh})^{\rho/(1-\rho)}\cdot\left(\frac{\alpha_f}{\alpha_h}\right)^{\rho/(1-\rho)} + w_f^{\rho/(1-\rho)} = (1+\tau_{hf})^{\rho/(1-\rho)}\cdot\left(\frac{\alpha_f}{\alpha_h}\right)^{\rho/(\rho-1)}\cdot w_f^{\frac{1+\rho}{1-\rho}} + w_f^{1/(1-\rho)}}$$

(4) Since the rich country's θ decreases from 0.5 to 0.25 it allocates a smaller fraction of its income on tradables. This will reduce the volume of trade, and hence reduce the gains from trade

(1) $\max\limits_{\{L_i^T\}} \; p_i^T Y_i^T - w_i L_i^T$ s.t. $Y_i^T = \alpha_i L_i^T$ $\qquad$ $\max\limits_{\{L_i^N\}} \; p_i^N Y_i^N - w_i L_i^N$ s.t. $Y_i^N = \beta_i L_i^N$

F.O.C $\qquad\qquad\qquad\qquad$ F.O.C

L_i^T: $p_i^T\alpha_i - w_i = 0 \Rightarrow p_i^T = \dfrac{w_i}{\alpha_i}$ $\qquad$ L_i^N: $p_i^N\beta_i - w_i = 0 \Rightarrow p_i^N = \dfrac{w_i}{\beta_i}$

Terms-of-Trade: $\dfrac{\text{Price of Exports}}{\text{Price of Imports}} = \dfrac{p_i^T}{(1+\tau_{ij})\,p_j^T} = \dfrac{w_i}{w_j}\cdot\dfrac{\alpha_j}{\alpha_i}\cdot\dfrac{1}{(1+\tau_{ij})}$