Lecture 3
Growth Model with Endogenous Savings:
Ramsey-Cass-Koopmans Model

Rahul Giri*

*Contact Address: Centro de Investigacion Economica, Instituto Tecnologico Autonomo de Mexico (ITAM). E-mail: rahul.giri@itam.mx

The Ramsey-Cass-Koopmans (Ramsey (1928), Cass (1965) and Koopmans (1965)) model is the standard infinite horizon neoclassical growth model. This model differs from the Solow model in one respect - it endogenizes the savings rate by explicitly modeling the consumer's decision to consume and save. This is done by adding a household optimization problem to the Solow model. This model has become the workhorse model not only for growth theory but also for macroeconomics.

1 Model Economy

1.1 Households

Consider an infinite horizon economy in continuous time with a large number of households, H. The size of each household grows at rate n. Each member of the household supplies 1 unit of labor at every point in time. The household rents whatever capital it owns to the firms. It has initial capital holdings of $K(0)/H$, where $K(0)$ is the initial capital stock of the economy. The households are also the owners of the firms. Therefore, they divide their labor income, rental income and profit income between consumption and savings. The utility function is given by

$$U = \int_{t=0}^{\infty} e^{-\rho t} u\left(C(t)\right) \frac{L(t)}{H} dt \quad ,$$

where $C(t)$ is the consumption of each member of the household at time t and $u\left(C\right)$ is the instantaneous utility function. $u\left(C\right)$ is defined on $\Re_+$ or $\Re_+\backslash\{0\}$ and it is strictly increasing, concave and twice differentiable, with derivatives $u'\left(C\right) > 0$ and $u''\left(C\right) < 0$ for all C in the interior of the domain of u. At this point in order to get rid of some notation, we can say that H is set of measure 1, i.e. there are an infinite number of households in the interval $(0, 1)$. The idea is the same - each household is too small to affect aggregates -, but it normalizes H to 1 and therefore aggregates are the same as averages. $L(t)$ is the total population of the economy. We assume that population within each household grows at rate n, starting with $L(0) = 1$. Therefore, $L(t) = \exp(nt)$. As a result the utility function can now be written as

$$U = \int_{t=0}^{\infty} e^{-(\rho - n)t} u\left(C(t)\right) dt \quad . \tag{1}$$

ρ is the subjective discount rate. It discounts streams of utility at t (from consumption at t) back to time 0. $(\rho - n)$ is the effective discount rate.

1.2 Firms

There are a large number of identical firms, as in the Solow model. We stick to the representative firm formulation. The aggregate production function of the economy is

$$Y(t) = F\left(K(t), A(t)L(t)\right) \quad , \tag{2}$$

As in the Solow model, Assumption 1 and 2 are imposed throughout. Taking $A(t)L(t)$ as effective labor we can write quantities per unit of effective labor. So $k = K/AL$ and $y = Y/AL = f(k)$. Factor markets are competitive, which implies that the rental rate of capital and the wage rate are given by:

$$R(t) = F_K\left(K(t), A(t)L(t)\right) = f'\left(k(t)\right) \quad , \tag{3}$$

$$w(t) = F_L\left(K(t), A(t)L(t)\right) = f\left(k(t)\right) - f'\left(k(t)\right)k(t) \quad . \tag{4}$$

With constant returns to scale firms do not make any profit, i.e. total revenue equals total cost.

1.3 Household's Budget Constraint

The representative household starts with a given stock of capital, $K(0)$, and then every period decides to save and consume. The savings/assets of the households are supplied to firms as capital. Then, budget constraint of the household can be written as

$$C(t)L(t) + \dot{K}(t) = w(t)L(t) + r(t)K(t) \quad . \tag{5}$$

The first term on the left hand side is the consumption expenditure and the second term is savings. On the right hand side the first term is labor income and the second term is rental income from capital. Remember, $r(t) = R(t) - \delta$. Household's total consumption, $C(t)L(t)$, is consumption per unit of effective labor as c times the effective labor AL. Similarly, the household's total labor income, wL, is wage rate w times effective labor AL. Since $k = K/AL$

$$\Rightarrow \dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right) \quad .$$

Substituting this in the budget constraint gives us the budget constraint in terms of effective labor units.

$$\dot{k} = \left(r(t) - n - g\right)k(t) + w(t) - c(t) \tag{6}$$

1.4 No-Ponzi Condition

The household budget constraint is not enough to get a sensible solution to the utility maximization problem. Notice that higher consumption leads to higher utility. And, the household could achieve higher and higher consumption levels by borrowing every period or having negative savings/assets. This could be done by borrowing more and more every period to service the pre-existing debt. This is what is called a *ponzi scheme*. We need to impose additional condition(s) to rule out ponzi schemes. There are two ways to go about this.

One is to impose a *natural debt limit*. This requires that the asset levels of the household do not turn so negative that the household cannot repay its debts even if henceforth it chooses zero consumption. Using Eq(6), suppose from time t onwards the household does not consume. Then, the natural debt limit for time t is

$$k(t) = -\int_t^\infty w(s) \exp\left(-\int_t^s (r(z) - n - g)\, dz\right) ds \quad ,$$

and its limiting version is

$$\lim_{t\to\infty} k(t) \geq \widehat{k} \equiv -\lim_{t\to\infty} \left[\int_t^\infty w(s) \exp\left(-\int_t^s (r(z) - n - g)\, dz\right) ds\right] \quad . \tag{7}$$

However, in economies with sustained growth $R(t)$ (and hence $r(t)$) is constant and hence $\widehat{k} = -\infty$. Therefore, the natural debt limit is not suitable. In such a case we impose the no ponzi scheme condition. To get to the condition let us start with the lifetime budget constraint of the consumer for some arbitrary $T > 0$.

$$\int_0^T C(t)L(t) \exp\left(\int_t^T r(s)ds\right) dt + K(T) = \int_0^T w(t)L(t) \exp\left(\int_t^T r(s)ds\right) dt + K(0) \exp\left(\int_t^T r(s)ds\right) \quad ,$$

where $K(T)$ is the asset position of the household at time T. This constraint states that the household's asset position at time T is given by its total income plus its initial assets minus the expenditures, all carried forward to date T units. Notice, that differentiating this expression with respect to T and dividing by $A(t)L(t)$ gives us Eq(6). Suppose the economy were to end at T, then it cannot be that the household has negative assets at this point in time. In other words, along with the lifetime budget constraint of the finite economy we also need to impose $K(T) \geq 0$, which is a *terminal value constraint*. The infinite economy case has the following version of the terminal value constraint:

$$\lim_{t\to\infty} \left[k(t) \exp\left(-\int_0^t (r(s) - n - g)\, ds\right)\right] \geq 0 \quad . \tag{8}$$

This condition is merely stating that the present discounted value of lifetime assets cannot be negative.

2 Equilibrium

Definition 1: A competitive equilibrium of the neoclassical growth model consists of paths of consumption, capital stock, wage rates, and rental rates of capital, $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$, such that the household maximizes its utility given its initial asset holdings (capital stock) $K(0) > 0$ and taking the path of prices $[w(t), R(t)]_{t=0}^{\infty}$ as given; firms maximize profits taking the time path of factor prices $[w(t), R(t)]_{t=0}^{\infty}$ as given; and factor prices $[w(t), R(t)]_{t=0}^{\infty}$ are such that all markets clear.

OR

Definition 2: A competitive equilibrium of the neoclassical growth model consists of paths of per capita consumption (in effective labor units), capital-labor ratio stock (in effective labor units), wage rates, and rental rates of capital, $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the factor prices $[w(t), R(t)]_{t=0}^{\infty}$ are given by (4) and (3) and the representative household maximizes (1) subject to (6) and (8) given its initial per capita asset holdings (capital-labor ratio) $k(0) > 0$ and factor prices $[w(t), R(t)]_{t=0}^{\infty}$.

The first definition of equilibrium is in terms of aggregate quantities and it does not impose any equilibrium relationships. The second definition is in terms of units of effective labor and it imposes equilibrium relationships to pin down factor prices.

2.1 Characterization of Equilibrium

Let us assume that preferences are given by:

$$u\left(C(t)\right) = \begin{cases} \frac{C(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \ , \\ \log C(t) & \text{if } \theta = 1 \ , \end{cases}$$

which implies that the utility function expressed in terms of consumption per unit of effective labor is given by:

$$U = \int_{t=0}^{\infty} e^{-(\rho-n-(1-\theta)g)t} u\left(c(t)\right) dt \quad . \tag{9}$$

Then the current value Hamiltonian for the utility maximization problem of the household is

$$\widehat{H}(t, k, c, \mu) = u\left(c(t)\right) + \mu(t)\left[w(t) + \left(r(t) - n - g\right) k(t) - c(t)\right] \quad ,$$

where the state variable is k and the control variable is c, and the current co-state variable is μ. The first-order conditions are

$$\widehat{H}_c(t,k,c,\mu) = 0 \Rightarrow u'\left(c(t)\right) - \mu(t) = 0 \quad , \tag{10}$$

$$\widehat{H}_k(t,k,c,\mu) = -\dot{\mu}(t) + (\rho - n - (1-\theta)g)\mu(t) \Rightarrow \mu(t)\left(r(t) - n - g\right) = -\dot{\mu}(t) + (\rho - n - (1-\theta)g)\mu(t) \quad , \tag{11}$$

$$\widehat{H}_\mu(t,k,c,\mu) = \dot{k}(t) \Rightarrow \dot{k}(t) = w(t) + \left(r(t) - n - g\right)k(t) - c(t) \quad . \tag{12}$$

Lastly, there is transversality condition:

$$\lim_{t\to\infty}\left[\exp\left(-(\rho - n - (1-\theta)g)t\right)\mu(t)k(t)\right] = 0 \quad . \tag{13}$$

Eq(10) implies that $u'\left(c(t)\right) = \mu(t)$. Differentiating this expression with respect to time and dividing both sides by $\mu(t)$ gives

$$\frac{u''\left(c(t)\right)c(t)}{u'\left(c(t)\right)}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)} \quad .$$

Let us make some substitutions in this equation. First, define the elasticity of marginal utility of consumption (expressed in effective labor units) as:

$$\epsilon_u\left(c(t)\right) = -\frac{u''\left(c(t)\right)c(t)}{u'\left(c(t)\right)} \quad . \tag{14}$$

Second, Eq(11) implies that

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\left(r(t) - \rho - \theta g\right) \quad . \tag{15}$$

Making these substitutions gives us the consumption *Euler equation.*

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\epsilon_u\left(c(t)\right)}\left(r(t) - \rho - \theta g\right) \quad . \tag{16}$$

The Euler equation states that consumption grows over time when $(\rho + \theta g)$ is less than the rate of return to capital. Notice that the elasticity of marginal utility is also the inverse of the *intertemporal elasticity of substitution*, which regulates the willingness of households to substitute consumption (or any other attribute that yields utility) over time.

Furthermore, integrating Eq(15) yields

$$\mu(t) = \mu(0)\exp\left(-\int_o^t (r(s) - \rho - \theta g)ds\right) \quad ,$$

$$\Rightarrow \mu(t) = u'\left(c(0)\right)\exp\left(-\int_0^t (r(s) - \rho - \theta g)ds\right) \quad .$$

Substituting this in the transversality condition and simplifying gives the following:

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (r(s)-n-g)ds\right)\right]=0 \quad .$$

Eq(3) implies that $r(t)=f'(k(t))-\delta$. Substituting this into the consumption Euler equation and the transversality condition gives us a complete characterization of the competitive equilibrium in terms of the path of the capital-labor ratio.

$$\frac{\dot{c}(t)}{c(t)}=\frac{1}{\epsilon_u(c(t))}\left(f'(k(t))-\delta-\rho-\theta g\right) \quad ,$$

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (f'(k(s))-\delta-n-g)ds\right)\right]=0 \quad .$$

2.2 Balanced Growth

But, what about balanced growth. Remember, balanced growth is characterized by constant rate of growth of output, constant capital-labor ratio and constant capital share in national income. These observations also imply that the rate of return on capital, $R(t)$ (and hence $r(t)$), is also constant. Balanced growth also requires that consumption and output also grow at a constant rate. At this point we need to clarify the importance of the assumption of the specific utility function we started with. Going back to the Euler equation

$$\frac{\dot{c}(t)}{c(t)}=\frac{1}{\epsilon_u(c(t))}(r(t)-\rho-\theta g) \quad .$$

When $r(t)\to r^*$, then $\dot{c}(t)/c(t)\to g_c$ is possible only if $\epsilon_u(c(t))\to\epsilon_u$, i.e. if the elasticity of marginal utility of consumption is asymptotically constant. Thus, balanced growth is possible only if the utility function has a asymptotically constant marginal utility of consumption.

In the neoclassical growth model, for the balanced growth path to exist, all technical change has to be asymptotically labor-augmenting and the intertemporal elasticity of substitution has to be asymptotically constant.

In case of our utility function, the elasticity of marginal utility of consumption, ϵ_u, is given by the constant θ. Therefore, the intertemporal elasticity of substitution is given by $1/\theta$. This utility function falls in the category of constant relative risk aversion utility functions (CRRA), which have the property that the Arrow-Pratt coefficient of relative risk aversion $-u''(c)c/u'(c)$ is constant. In case of the utility function we have considered the coefficient of relative risk aversion is θ. Thus, the elasticity of intertemporal substitution is the inverse of the coefficient of relative risk aversion. Notice that when θ is zero, the preferences are linear and the agents are risk neutral, and therefore infinitely willing to substitute consumption over time. Whereas when $\theta \to \infty$, agents are infinitely risk averse and infinitely unwilling to substitute consumption over time.

Using the CRRA feature of the preferences gives us the following version of the consumption Euler equation.

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left(r(t) - \rho - \theta g\right) \quad .$$

Since, on the balanced growth path, $\dot{c}(t) = 0$, it implies that the interest rate $r^* = \rho + \theta g$. Given that $r(t) = f'\left(k(t)\right) - \delta$, the equation pinning down k^* on the balanced growth path is

$$f'\left(k^*\right) = \rho + \delta + \theta g \quad . \tag{17}$$

Again, on the balanced growth path, $\dot{k}(t) = 0$. Due to CRS, $w(t) = f\left(k(t)\right) - f'\left(k(t)\right)k(t)$. Substituting these in the law of motion for capital gives us the consumption on the balanced growth path.

$$c^* = f\left(k^*\right) - (n + g + \delta)k^* \quad . \tag{18}$$

The transversality condition, after substituting for $f'\left(k(t)\right)$, becomes

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (\rho - (1-\theta)g - n)ds\right)\right] = 0 \quad , \tag{19}$$

which can hold only if the integral in the exponent goes to minus infinity, that is only if $\rho - (1 - \theta)g - n > 0$. Thus, to ensure a well defined solution to the household maximization and to the competitive equilibrium we have to assume that $\rho - n > (1 - \theta)g$. This guarantees that $r^* > g + n$, which is essential to ensure that households do not achieve infinite utility.

As in the Solow model, the aggregate output and the aggregate capital grow at rate $(n + g)$, which from Eq(18) implies that aggregate consumption also grows at $(n + g)$. The per worker output, capital and consumption grow at rate g. Thus, endogenizing the saving decision of the households does not affect the growth rate of the economy, and the exogenous growth in technology still remains the source of growth. The savings rate, $(y - c)/y$, is constant because y and c are constant. The steady-state capital labor ratio, k^*, is endogenous and depends on the instantaneous utility function of the representative household, since now k^* is a function of θ. Since households face an upward sloping consumption profile, their willingness to substitute consumption today for consumption tomorrow determines how much they accumulate and thus the equilibrium effective capital-labor ratio.

3 Transitional Dynamics

Unlike the Solow model where only one equation governed the transitional dynamics of the economy, in the neoclassical growth model we have two equations.

$$\dot{k}(t) = f\left(k(t)\right) - c(t) - (n + g + \delta)k(t) \quad ,$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left(f'\left(k(t)\right) - \delta - \rho - \theta g\right) \quad .$$

Furthermore, we have an initial condition $k(0) > 0$ and a boundary condition at infinity given by the transversality condition.

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (f'\left(k(s)\right) - \delta - n - g)ds\right)\right] = 0 \quad .$$

The next two figures show the dynamics of c and k in a $c - k$ plane. The first figure shows the dynamics of c. When $\dot{c} = 0$, $k = k^*$, which is the level of effective capital-labor ratio on the balanced growth path. Thus, only at k^* is the effective per capita consumption constant. This is captured by the vertical line $\dot{c} = 0$. At $\dot{c} = 0$, $f'\left(k^*\right) = \delta + \rho + \theta g$. Thus, when $k > k^*$, $f'(k) < \delta + \rho + \theta g$, and so $\dot{c}$ is negative, implying that c falls. On the other hand, when $k < k^*$, $f'(k) > \delta + \rho + \theta g$, and so $\dot{c}$ is positive, implying that c rises.

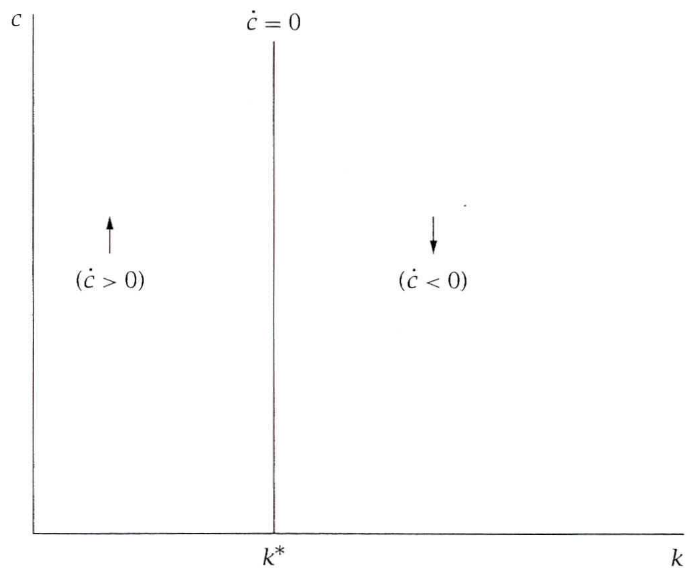

Figure 1: The dynamics of c

The dynamics of k works in the same manner as in the Solow model. $\dot{k}$ equals actual investment (output minus consumption) $f\left(k(t)\right) - c(t))$ and break even investment $(n + g + \delta)k(t))$. For a given k, the level of c that ensures that $\dot{k} = 0$ is given by $f(k) - (n + g + \delta)k$, which means that consumption equals the difference between output and break-even investment. This is depicted in the next figure as the $\dot{k} = 0$ curve. c is increasing in k till $f'(k) = (n+g+\delta)$ (golden-rule level of k), and is then decreasing in k. When c exceeds the level that yield $\dot{k} = 0$, i.e. $c > f(k) - n + g + \delta)k$, k falls and when c is less than this level k rises. For k sufficiently large, break-even investment exceeds total output, and so $\dot{k}$ is negative for all positive values of c.

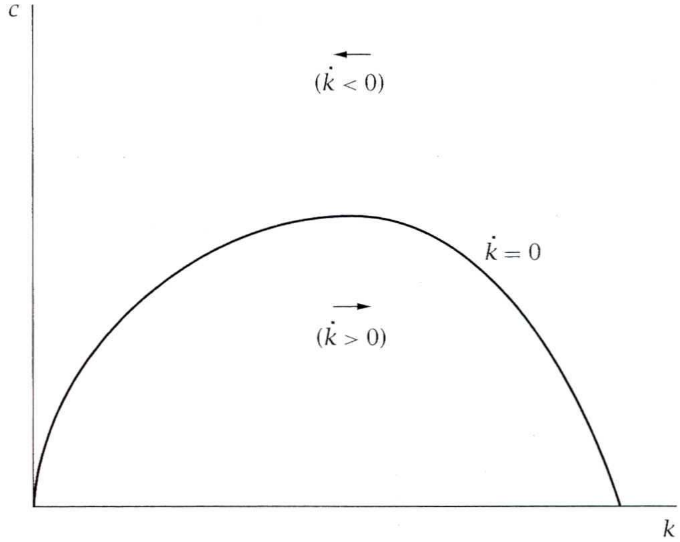

Figure 2: The dynamics of k

The next graph shows the transition dynamics for an economy, starting with a given $k(0)$. Given the directions of the movements of c and k, there exists a unique stable arm tending to the steady state. In other words, starting with a $k(0) > 0$, there exists a unique $c(0)$ such that the consumption path implied by the Euler equation takes the economy to the steady state values of c and k - (c^*, k^*). Furthermore, this path is unique. For the same given $k(0)$, all other paths have a $c(0)$, which results in divergent behavior. For example, the path which starts with $c'(0)$ sees c rising and and k ultimately falling. For the two dynamic equations to be satisfied c must continue to rise and k must become negative. But, this is not possible because once k becomes zero, y also becomes zero and hence c becomes zero. On the other hand, when the economy starts with a $c''(0)$, k eventually exceeds the golden-rule level which implies that $f'(k) < (n + g + \delta)$. This in turn will cause the transversality condition to not hold because

$$\lim_{t \to \infty} \left[k(t) \exp \left(- \int_0^t (f'(k(s)) - \delta - n - g) ds \right) \right] \to \infty \quad ,$$

which means that households discounted lifetime income becomes infinitely large and the household can raise consumption levels and hence raise its utility. Thus, all paths other than the stable arm/saddle path are not equilibrium paths because they violate one or more of the equilibrium conditions.

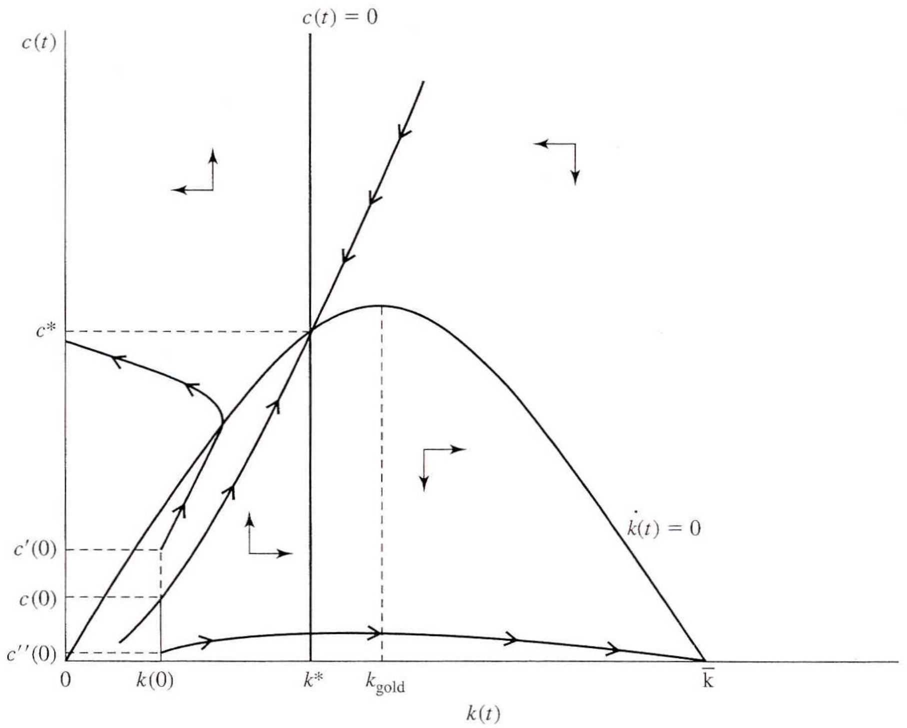

Figure 3: Transitional dynamics in the neoclassical growth model

An important difference between the Solow model and the neoclassical growth model is that a balanced growth path with a capital stock above the golden-rule level is not possible in the neoclassical model. In the Solow model, a sufficiently high savings rate causes the economy to reach a balanced growth path with the property that there are feasible alternatives that involve higher consumption at every t. However, in the neoclassical growth model, savings is an implication of household optimizing behavior. As a result, it cannot be an equilibrium for the economy to follow a path where higher consumption can be attained in every period. This can be seen in the figure above. If, to start with, $k(0)$ is higher than the golden-rule level, $c(0)$ is higher than the level needed to keep constant and $\dot{k} < 0$. Therefore k will fall and gradually approach k^*, which is below the golden rule level. Because k^* is is the result of optimal household behavior it is called the *modified golden-rule* capital stock.

4 The Effects of a Fall in Discount Rate

In this section we would like to analyze the case analogous to the case of an increase in savings rate in the Solow economy. So, consider a neoclassical economy that is on the balanced growth path, and suppose there is a fall in the discount rate, ρ. Since ρ governs how much households value future consumption relative to present consumption, it is the closest analouge in this model to a rise in the savings rate in the Solow model. Since the households are forward looking we also assume that this change is unexpected.

ρ enters only the consumption Euler equation. The $\dot{c} = 0$ locus is characterized by the condition that $f'(k^*) = \rho + \theta g$. Due to diminishing returns, a fall in ρ results in an increase in k^*, which shifts the $\dot{c} = 0$ to the right. This is shown in the figure below. At the time of the change in ρ, the existing value of k, which is k^* on the old balanced growth path, cannot change discontinuously since it is governed by the history of the economy. c, on the other hand, can have a discontinuous change. Therefore, at the instant of the change in ρ, c falls to point A so that economy is on the saddle path. Thereafter c and k rise gradually to their new balanced growth path values, which are higher than the old balanced growth path values. Thus, the effects of a fall in the discount rate are similar to the effects of a rise in saving rate in the Solow model with capital stock below the golden-rule level. In both cases, k rises gradually to its new higher level, and in both cases c initially falls but then rises above the level it started at. While k is gradually rising there is a temporary increase in the growth rates of capital and output per capita. The only difference is that in the neoclassical economy when ρ falls the fraction of income saved is not constant during the transition to the new balanced growth path.

5 The Effects of Government Purchases

An important component of final demand in most countries is government expenditure. In the U.S., about 20% of total output is purchased by the government. So, how does the presence of a government effect the results of the neoclassical model?

Suppose, the government buys $G(t)$ units of the final good, measured in units of effective labor. Government purchases are assumed to not affect utility from private consumption. This is the case if government devotes the goods to some activity that does not affect utility at all, or if utility is the sum of utility from private consumption and utility from government provided

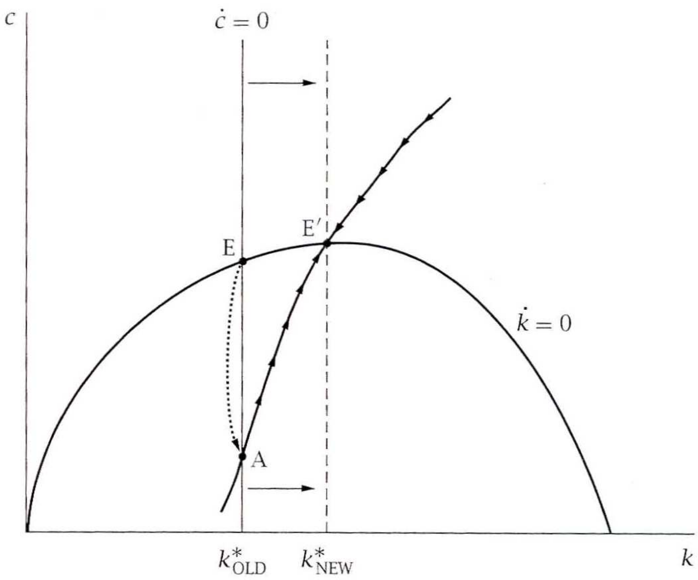

Figure 4: The effect of a fall in the discount rate

goods. Similarly, the purchases are assumed not affect future output, i.e. they are devoted to public consumption rather than public investment. Government purchases are financed by lump-sum taxes of amount $G(t)$ per unit of effective labor. Thus, the government always runs a balanced budget.

Investment, is now, the difference between output and the sum of private consumption and government consumption. Thus, the law of motion for capital in units of effective labor is given by:

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g + \delta)k(t) \quad ,$$

The presence of $G(t)$ shifts the $\dot{k} = 0$ locus down. For a given level of k the higher the fraction of output purchased by the government the lower the amount left for private consumption, and hence $c(t)$ is lower. Since the consumption Euler equation is driven by preferences, which are unchanged, this remains unchanged.

5.1 Permanent Increase in Government Purchases

Suppose the economy is on a balanced growth path with $G(t)$ constant at G_L, and then there is an unexpected, permanent increase in G to G_H. This will shift the $\dot{k} = 0$ locus down by the amount of increase in G. This is shown in the figure that follows. The effect is that c jumps down to the

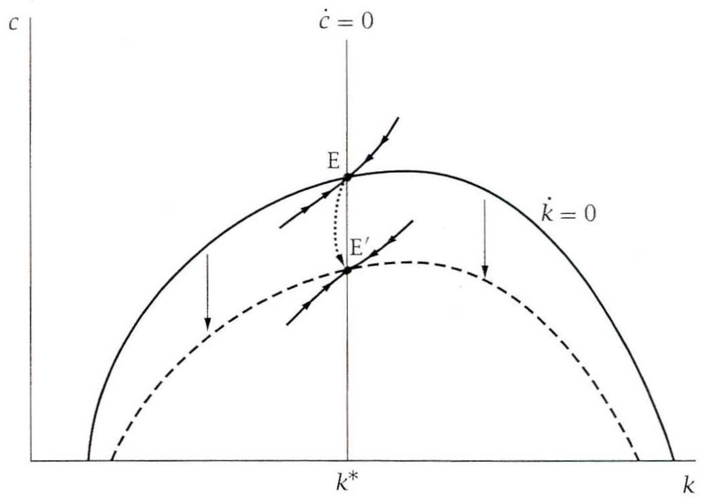

Figure 5: The effects of a permanent increase in government purchases

new saddle path (from E to E'), by exactly the same amount as the increase in G. The economy is immediately on its new balanced growth path.

Intuitively, an increase in government expenditure (or taxes) reduces households lifetime wealth, and because this change is permanent there is no way for the households to change their consumption profile so as to increase their utility. Thus, the immediate fall in consumption is equal to the full amount of the increase in government purchases. The capital stock and real interest rate are unaffected.

5.2 Temporary Increase in Government Purchases

Now let us consider a case of an unanticipated increase in G that is expected to be temporary. Suppose, households know the exact date when government purchases would go back to their old level, G_L. In this case, c does not fall by the full amount of the increase in government purchases. This is because the households know that the change in government purchases will be reversed at a future date. Since, the households know the timeline of these changes and since they are forward looking, the adjustment in c when G goes back to G_L will be smooth.

When the unexpected increase in G takes place (from G_L to G_H), c jumps to the value which ensures that dynamics implied by the consumption Euler equation and the law of motion for capital, with $G = G_H$, take the economy to back to the old saddle path at the time when G falls

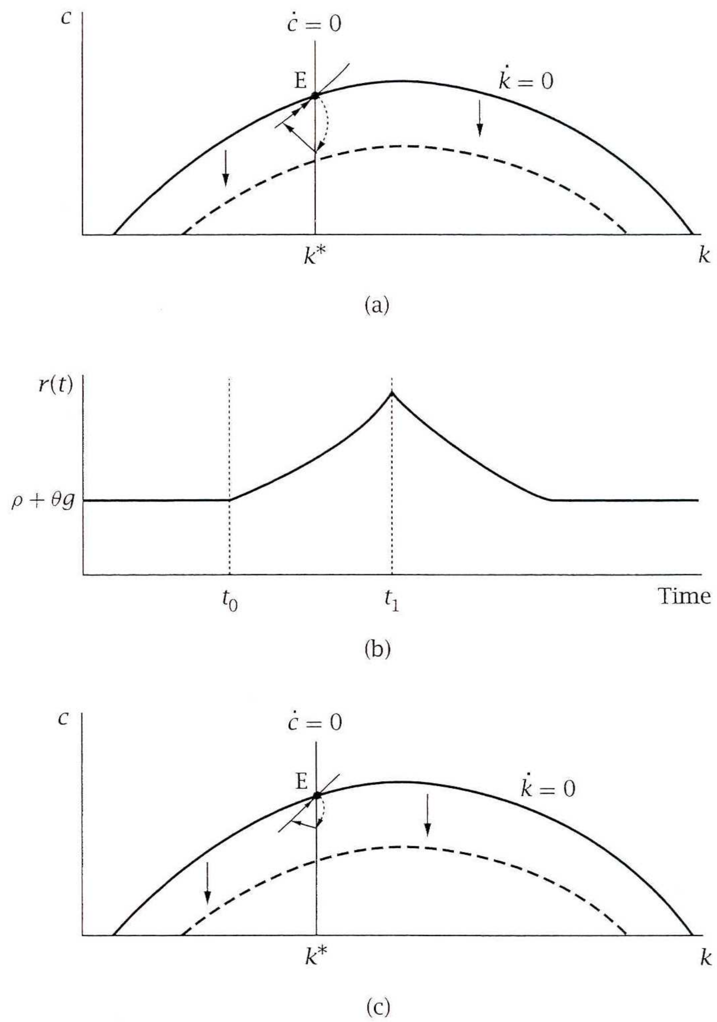

Figure 6: The effects of a temporary increase in government purchases

back to its initial level of G_L. Thereafter, the economy moves along the old saddle path to the old balanced growth path. This is depicted in the figure above.

Panel (a) shows the case where the increase in G is relatively long lasting, and therefore c falls by most of the amount of the increase in G. As the economy returns to G_L households increase their consumption and reduce their capital holdings in anticipation of the fall in G. Panel (b) shows the behavior of $r = f'(k) - \delta$. r rises gradually during the period when G is high and then falls gradually to its initial level. t_0 denotes the time of the increase in G, and t_1 denotes the time of its return to its initial value. Panel (c) shows the case of a short-term increase in G. In this case, households change their consumption by a smaller amount, choosing to pay for the higher taxes largely from their savings. Because of the short-term increase in G the effects on k and r are small.

6 References

1. Cass, David (1965), "Optimum Growth in an Aggregate Model of Capital Accumulation", Review of Economic Studies 32: 233-240.

2. Koopmans, Tjalling C. (1965), "On The Concept of Optimal Economic Growth", in The Econometric Approach to Development Planning, Amsterdam: North-Holland, pp. 225-195.

3. Ramsey, Frank (1928), "A Mathematical Theory of Saving", Economic Journal 38: 543-559.