

The Theorem of the Maximum

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Correspondences

- In what follows, all sets will be assumed to be topological spaces that satisfy the Hausdorff separation property (distinct points have disjoint open neighborhoods).
- We say that Γ is a **correspondence** from X to Y , denoted $\Gamma : X \rightrightarrows Y$, if to each $x \in X$ it associates a subset $\Gamma(x) \subset Y$.
- For example, let $X = Y = \mathbb{R}$, and $\Gamma(x) = \{y \in \mathbb{R} : 0 \leq y \leq x^2\}$.
- A correspondence is a convenient way to express the constraints that affect the decision variables in an optimization problem.
- We will also assume that all correspondences are **nonempty-valued**: for all $x \in X$, $\Gamma(x) \neq \emptyset$.

Continuity of correspondences

- We say that Γ is **upper hemicontinuous (uhc)** at $x \in X$ if, for each $U \subset Y$ open such that $\Gamma(x) \subset U$, there is an open neighborhood V of x such that, for all $x' \in V$, $\Gamma(x') \subset U$.
- We say that Γ is **lower hemicontinuous (lhc)** at $x \in X$ if, for each $U \subset Y$ open such that $\Gamma(x) \cap U \neq \emptyset$, there is an open neighborhood V of x such that, for all $x' \in V$, $\Gamma(x') \cap U \neq \emptyset$.
- We say that Γ is **continuous** at $x \in X$ if it is both upper and lower hemicontinuous at x .
- We say that Γ is upper hemicontinuous (resp. lower hemicontinuous, continuous) if it is uhc (resp. lhc, continuous) at each point of its domain.
- Note that, according to these definitions, if Γ is **constant** ($\exists A \subset Y, \forall x \in X, \Gamma(x) = A$), then it is continuous at all points.

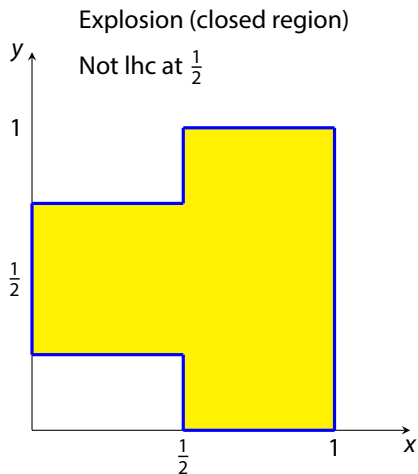
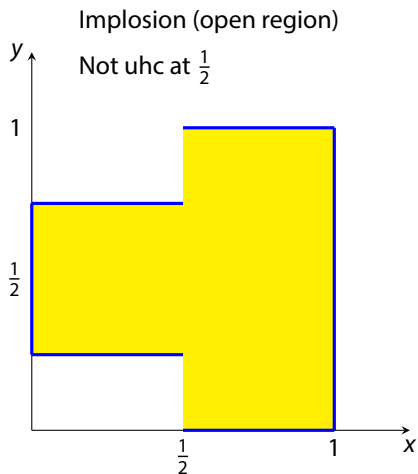
Intuition of upper hemicontinuity

- In the examples that follow, let $X = Y = [0, 1]$.
- Upper hemicontinuity prevents an **implosion** of the correspondence.
- For example, if $\Gamma(x) = [1/4, 3/4]$ for all $x \neq 1/2$, and $\Gamma(1/2) = \{1/2\}$, then Γ fails to be uhc at $x = 1/2$, because the correspondence implodes at this point. However, Γ is lhc at all points.
- In general, an implosion results in an **open region** in the graph of the correspondence.
- For example, let $\Gamma(x) = [1/4, 3/4]$ for all $x \leq 1/2$, and $\Gamma(x) = [0, 1]$ for all $x > 1/2$. Again, Γ fails to be uhc at $x = 1/2$, but it is lhc all points.

Intuition of lower hemicontinuity

- In the examples that follow, let $X = Y = [0, 1]$.
- Lower hemicontinuity prevents an **explosion** of the correspondence.
- For example, if $\Gamma(x) = [1/4, 3/4]$ for all $x \neq 1/2$, and $\Gamma(1/2) = [0, 1]$, then Γ fails to be lhc at $x = 1/2$, because the correspondence explodes at this point. However, Γ is uhc at all points.
- In general, an explosion results in a **closed region** in the graph of the correspondence.
- For example, let $\Gamma(x) = [1/4, 3/4]$ for all $x < 1/2$, and $\Gamma(x) = [0, 1]$ for all $x \geq 1/2$. Again, Γ fails to be lhc at $x = 1/2$, but it is uhc all points.

Intuition: representation




Inverse mappings

- Given a set $A \subset Y$, the **upper inverse of A under Γ** is the set:

$$\Gamma^+(A) \doteq \{x \in X : \Gamma(x) \subset A\}$$

- Given a set $A \subset Y$, the **lower inverse of A under Γ** is the set:

$$\Gamma^-(A) \doteq \{x \in X : \Gamma(x) \cap A \neq \emptyset\}$$

 The following identities are easy to verify. For all $A \subset Y$:

$$\Gamma^+(A^c) = [\Gamma^-(A)]^c, \quad \Gamma^-(A^c) = [\Gamma^+(A)]^c$$

Characterizations of hemicontinuity

Proposition (Characterization of uhc)

The following statements are equivalent:

- 1 $\Gamma : X \rightarrow Y$ is uhc.
- 2 For all $U \subset Y$ open, $\Gamma^+(U)$ is open in X .
- 3 For all $F \subset Y$ closed, $\Gamma^-(F)$ is closed in X .

Proposition (Characterization of lhc)

The following statements are equivalent:

- 1 $\Gamma : X \rightarrow Y$ is lhc.
- 2 For all $U \subset Y$ open, $\Gamma^-(U)$ is open in X .
- 3 For all $F \subset Y$ closed, $\Gamma^+(F)$ is closed in X .

Proof of the characterizations

- We will only prove the uhc result, since the other is similar.
- The equivalence between (2) and (3) derives from the identities regarding upper and lower inverses and complementary sets.
- (1) \Rightarrow (2): $x \in \Gamma^+(U)$ implies $\Gamma(x) \subset U$, so by uhc there is $V \ni x$ open such that $\forall x' \in V, \Gamma(x') \subset U$. This means that $V \subset \Gamma^+(U)$, and so this latter set is open.
- (2) \Rightarrow (1): This is implied a fortiori. Given $x \in X$ and U open with $\Gamma(x) \subset U$, then $\Gamma^+(U)$ is an open neighborhood of x that satisfies the requirement in the definition of uhc.

Semicontinuous functions

- A real-valued function $f : X \rightarrow \mathbb{R}$ is **upper semicontinuous (usc)** at $x \in X$ if, for all $\varepsilon > 0$, there is an open neighborhood V of x such that, for all $x' \in V$,
 $f(x') \leq f(x) + \varepsilon$.
- A real-valued function $f : X \rightarrow \mathbb{R}$ is **lower semicontinuous (lsc)** at $x \in X$ if, for all $\varepsilon > 0$, there is an open neighborhood V of x such that, for all $x' \in V$,
 $f(x') \geq f(x) - \varepsilon$.
- The function f is continuous at x if, and only if, it is both usc and lsc at x .
- The function is upper semicontinuous (lower semicontinuous) if it is usc (lsc) at each point in its domain.
- **Duality property:** a function f is usc if, and only if, $-f$ is lsc.

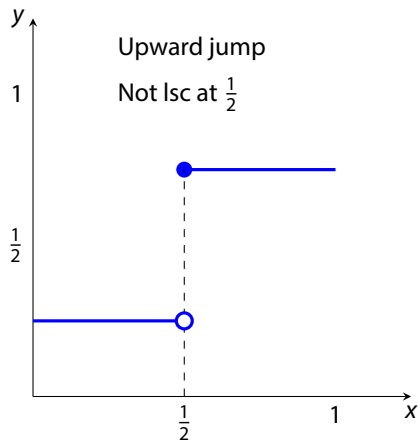
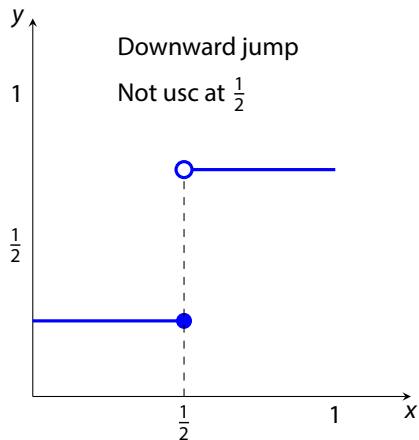
Intuition of upper semicontinuity

- In the examples that follow, let $X = Y = [0, 1]$.
- Upper semicontinuity **prevents downward jumps** of the function, but **allows upward jumps**.
- For example, let $f(x) = 1/2$ for all $x \neq 1/2$, and $f(1/2) = 0$. Then f fails to be usc at $x = 1/2$, but it is lsc at all points.
- In general, a downward jump leaves an **open region** below the graph of the function.
- Let Γ be the (hypograph) correspondence defined by $\Gamma(x) = [0, f(x)]$. Then Γ fails to be uhc at $1/2$, but is lhc at all points.
- For another example, let $f(x) = 1/4$ for all $x \leq 1/2$, and $f(x) = 3/4$ for all $x > 1/2$, then f fails to be usc at $x = 1/2$, because the function has a downward jump at $1/2$. But f is lsc at all points.
- More formally, $A \subset X$ is closed if, and only if, the indicator function $I_A(x)$ is usc. (The indicator function takes the value 1 if $x \in A$ and 0 otherwise.)

Intuition of lower semicontinuity

- In the examples that follow, let $X = Y = [0, 1]$.
- Lower semicontinuity **prevents upward jumps** of the function, but **allows downward jumps**.
- For example, let $f(x) = 1/2$ for all $x \neq 1/2$, and $f(1/2) = 1$. Then f fails to be lsc at $x = 1/2$, but it is usc at all points.
- In general, an upward jump leaves a **closed region** below the graph of the function.
- Let Γ be the (hypograph) correspondence defined by $\Gamma(x) = [0, f(x)]$. Then Γ fails to be lhc at $1/2$, but is uhc at all points.
- For another example, let $f(x) = 1/4$ for all $x < 1/2$, and $f(x) = 3/4$ for all $x \geq 1/2$, then f fails to be lsc at $x = 1/2$, because the function has an upward jump at $1/2$. But f is usc at all points.
- More formally, $A \subset X$ is open if, and only if, the indicator function $I_A(x)$ is lsc.

Intuition: representation



Semicontinuity and Hemicontinuity

Proposition

Given $f : X \rightarrow \mathbb{R}$, define the *hypograph correspondence*, $\Gamma : X \rightrightarrows \mathbb{R}$, and the *epigraph correspondence*, $\Phi : X \rightrightarrows \mathbb{R}$, by:

$$\Gamma(x) = \{y \in \mathbb{R} : y \leq f(x)\}, \quad \Phi(x) = \{y \in \mathbb{R} : y \geq f(x)\}.$$

Then we have:

- 1 f usc at $x \iff \Gamma$ uhc at $x \iff \Phi$ lhc at x .
- 2 f lsc at $x \iff \Gamma$ lhc at $x \iff \Phi$ uhc at x .

The proof is left as an exercise.

Characterizations of semicontinuity

Proposition (Characterization of usc)

The following statements are equivalent:

- 1 $f : X \rightarrow \mathbb{R}$ is usc.
- 2 For all $a \in \mathbb{R}$, $\{x \in X : f(x) \geq a\}$ is closed in X .
- 3 For all $a \in \mathbb{R}$, $\{x \in X : f(x) < a\}$ is open in X .

Proposition (Characterization of lsc)

The following statements are equivalent:

- 1 $f : X \rightarrow \mathbb{R}$ is lsc.
- 2 For all $a \in \mathbb{R}$, $\{x \in X : f(x) \leq a\}$ is closed in X .
- 3 For all $a \in \mathbb{R}$, $\{x \in X : f(x) > a\}$ is open in X .

Proof of the characterizations

- We will only prove the usc result, since the other follows by [duality](#).
- The equivalence between (2) and (3) derives from complementation.
- (1) \Rightarrow (3): Let $a \in \mathbb{R}$. If no x satisfies $f(x) < a$ we are done. Otherwise, let $x \in X$ satisfy $f(x) < a$. Let $\varepsilon > 0$ satisfy $f(x) < a - \varepsilon$, then by usc of f there is $V \ni x$ open such that $\forall x' \in V, f(x') \leq f(x) + \varepsilon < a$. This shows that $\{x \in X : f(x) < a\}$ is open.
- (3) \Rightarrow (1): Let $x \in X$ and $\varepsilon > 0$. The set $\{x' \in X : f(x') < f(x) + \varepsilon\}$ is an open neighborhood of x that satisfies the requirement in the definition of usc.

Semicontinuous functions with extended real values

- It is convenient to allow lsc functions to take also the value $+\infty$, and usc functions to take the value $-\infty$.
- In this case, the requirement (in terms of ε) in the respective definitions only applies to $x \in X$ for which $f(x) \in \mathbb{R}$.
- With these extended definitions, the previous characterizations (in which $a \in \mathbb{R}$) remain valid for such functions.
- The reason for those definitions is the following: Given a collection $(f_i)_{i \in I}$ of continuous functions, define $g(x) = \sup \{f_i(x) : i \in I\}$ and $h(x) = \inf \{f_i(x) : i \in I\}$. Then it may be checked that $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lsc and $h : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is usc in the extended sense we have defined.
- If the topological space X satisfies a regularity property (which all metric spaces satisfy), then it is also true that any (extended) lsc function g is the pointwise supremum over all continuous functions f such that $f \leq g$. Analogously, any (extended) usc function h is the pointwise infimum over all continuous functions f such that $f \geq h$.

Examples of semicontinuous functions with extended real values

- Let $X = [0, 1]$, and define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1/x, & \text{if } x > 0; \\ 0, & \text{if } x = 0. \end{cases}$$

- Then f is lsc and takes only real values.
- For all $i \in (0, 1)$, let $f_i(x) = x/i^2$ if $x \leq i$, and $f_i(x) = 1/x$ if $x > i$. Then each f_i is continuous, and $f(x) = \sup \{f_i(x) : i \in (0, 1)\}$.
- Define now $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ by:

$$g(x) = \begin{cases} 1/x, & \text{if } x > 0; \\ +\infty & \text{if } x = 0. \end{cases}$$

- Then g is lsc in the extended sense we have defined.
- For all $i \in (0, 1)$, let $g_i(x) = 1/i$ if $x \leq i$, and $g_i(x) = 1/x$ if $x > i$. Then each g_i is continuous, and $g(x) = \sup \{g_i(x) : i \in (0, 1)\}$.


Maxima and minima over compact sets

This generalizes Weierstrass's Theorem for continuous functions.

Theorem

Let $f : X \rightarrow \mathbb{R}$, and let $A \subset X$ be a compact (nonempty) set.

- 1 If f is usc, then $f(A) \subset \mathbb{R}$ has a maximum value, ie, there is $x^* \in A$ such that, for all $x \in A$, $f(x^*) \geq f(x)$.
- 2 If f is lsc, then $f(A) \subset \mathbb{R}$ has a minimum value, ie, there is $x^* \in A$ such that, for all $x \in A$, $f(x^*) \leq f(x)$.

 Note that the statement for lsc follows from the one for usc applied to $-f$ (ie, by duality).

Proof of the usc statement

- Let $a = \sup f(A) \in (-\infty, +\infty]$.
- Then, whether or not a is finite, there is a sequence $(a_n)_{n \in \mathbb{N}}$, such that, $\forall n \in \mathbb{N}$, $a_n \in f(A)$, $a_n \leq a_{n+1}$, and $a_n \uparrow a$ (this sequence might be constant from some point on).
- For $n \in \mathbb{N}$, define $A_n = \{x \in X : f(x) < a_n\}$.
- Note that $n_1 < n_2 \Rightarrow A_{n_1} \subset A_{n_2}$.
- By usc of f , each A_n is open.
- If the collection $(A_n)_{n \in \mathbb{N}}$ covers A , then by compactness of this set there is a finite subcover. Let n^* be the highest index in this subcover, then A_{n^*} covers A , so that, for all $x \in A$, $f(x) < a_{n^*}$, which contradicts $a_{n^*} \in f(A)$.
- Therefore, there exists $x^* \in A \setminus \bigcup_{n \in \mathbb{N}} A_n$. This means that, for all n , $a_n \leq f(x^*) \leq a$. Hence, $f(x^*) = a$, and thus $a \in \mathbb{R}$.

Maxima of lsc functions

Lemma (lsc of the value function)

Assume:

- 1 $f : X \times Y \rightarrow \mathbb{R}$ is a lsc function.
- 2 $\Gamma : X \rightrightarrows Y$ is a lhc correspondence (with nonempty values).

Define

$$v(x) = \sup \{f(x, y) : y \in \Gamma(x)\}.$$

Then $v : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc function.

Proof of lsc of the value function

- Let $a \in \mathbb{R}$, and $A = \{x \in X : v(x) > a\}$. We must show that A is open.
- If $A = \emptyset$ we are done. Otherwise, let $x \in A$.
- Now, $v(x) > a$ implies that there is $y \in \Gamma(x)$ such that $f(x, y) > a$.
- By lsc of f , there are open neighborhoods $U \ni x$ and $V \ni y$ such that, for all $x' \in U$ and $y' \in V$, $f(x', y') > a$.
- By lhc of Γ , $\Gamma^-(V)$ is an open neighborhood of x .
- So $U' = U \cap \Gamma^-(V)$ is an open neighborhood of x , and for all $x' \in U'$ there is $y' \in \Gamma(x')$ such that $f(x', y') > a$, which implies $v(x') > a$, ie $x' \in A$.
- We conclude $x \in U' \subset A$, and therefore A is open.

Maxima of usc functions

Lemma (usc of the value function)

Assume:

- 1 $f : X \times Y \rightarrow \mathbb{R}$ is an usc function.
- 2 $\Gamma : X \rightrightarrows Y$ is an uhc correspondence with (nonempty and) compact values.

Define

$$v(x) = \max \{f(x, y) : y \in \Gamma(x)\}.$$

Then $v : X \rightarrow \mathbb{R}$ is an usc function.

Proof of usc of the value function

- We know that $v(x)$ is the maximum because of the generalization of Weierstrass's Theorem to usc functions. Let $a \in \mathbb{R}$, and $A = \{x \in X : v(x) < a\}$. We must show that A is open.
- If $A = \emptyset$ we are done. Otherwise, let $x \in A$, so that $v(x) < a$.
- Let $\varepsilon > 0$ satisfy $v(x) < a - \varepsilon$. Thus, $\forall y \in \Gamma(x), f(x, y) < a - \varepsilon$.
- By usc of f , given $y \in \Gamma(x)$, there are open neighborhoods $V_y \ni y$ and $U_y \ni x$ such that, for all $x' \in U_y$ and $y' \in V_y, f(x', y') < a - \varepsilon$.
- The collection $(V_y)_{y \in \Gamma(x)}$ is an open cover of the compact set $\Gamma(x)$, so by compactness there is a finite subcover, indexed by $\{y_1, y_2, \dots, y_m\}$.
- Let V be the union of this finite subcover. By uhc of $\Gamma, \Gamma^+(V)$ is an open neighborhood of x .
- Let $U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_m} \cap \Gamma^+(V)$. Then $\forall x' \in U, \Gamma(x') \subset V$ and $\forall y' \in \Gamma(x'), f(x', y') < a - \varepsilon$, so $v(x') \leq a - \varepsilon < a$.
- So $x \in U \subset A$, and therefore A is open.

Maxima of continuous functions

Theorem (The Theorem of the Maximum)

Assume:

- ① $f : X \times Y \rightarrow \mathbb{R}$ is a continuous function.
- ② $\Gamma : X \rightrightarrows Y$ is a continuous correspondence with (nonempty and) compact values.

Define

$$v(x) = \max \{f(x, y) : y \in \Gamma(x)\}.$$

Define also

$$G(x) = \arg \max \{f(x, y) : y \in \Gamma(x)\} = \{y \in \Gamma(x) : f(x, y) = v(x)\}.$$

Then $v : X \rightarrow \mathbb{R}$ is a continuous function and $G : X \rightrightarrows Y$ is an uhc correspondence with (nonempty and) compact values.

Proof of the Theorem of the Maximum (I)

- Continuity of v is a consequence of the previous lemmas.
- Let $x \in X$. We claim that $G(x)$ is closed. Let $y \notin G(x)$. If $y \notin \Gamma(x)$, then we know y has an open neighborhood disjoint from $G(x) \subset \Gamma(x)$, because the latter set is compact (hence closed).
- Assume $y \in \Gamma(x) \setminus G(x)$, so that $f(x, y) < v(x)$. By continuity (usc) of f in y , the set $U = \{y' \in Y : f(x, y') < v(x)\}$ is open, it contains y , and $U \cap G(x) = \emptyset$. This shows that $G(x)$ is closed, and therefore compact.
- We now show uhc of G at x . If $G(x) = \Gamma(x)$, then uhc of G follows from the same property of Γ .
- Assume, therefore, that $\Gamma(x) \setminus G(x) \neq \emptyset$.
- For each $y \in \Gamma(x) \setminus G(x)$, we have $f(x, y) < v(x)$, so by continuity of $v - f$ there are open neighborhoods $V_y \ni y$ and $W_y \ni x$ such that, for all $x' \in W_y$ and $y' \in V_y$, $f(x', y') < v(x')$.

Proof of the Theorem of the Maximum (II)

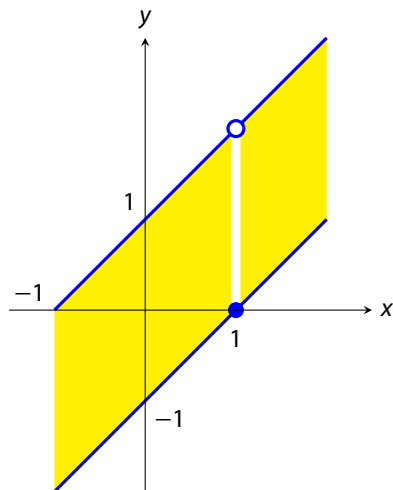
- Let U be an open neighborhood of $G(x)$. Then U together with the collection $\{V_y : y \in \Gamma(x) \setminus G(x)\}$ is an open cover of the compact set $\Gamma(x)$, so by compactness there is a finite subcover.
- Now U must be part of this finite subcover, since for all y , V_y is disjoint from $G(x)$, which is nonempty. Let $\{y_1, y_2, \dots, y_m\}$ be the indices corresponding to the remaining sets in the finite subcover.
- Let V be the union of this finite subcover. By uhc of Γ , $\Gamma^+(V)$ is an open neighborhood of x .
- Let $W = W_{y_1} \cap W_{y_2} \cap \dots \cap W_{y_m} \cap \Gamma^+(V)$.
- Let $x' \in W$, so $G(x') \subset \Gamma(x') \subset V$. If $y' \in \Gamma(x')$ and, for some i , $y' \in V_{y_i}$, then, since $x' \in W_{y_i}$, we have $f(x', y') < v(x')$. This means that $y' \notin G(x')$. Therefore $G(x') \subset U$. This concludes the proof of uhc of G .

Intuition of the lower semicontinuity of the maximum values

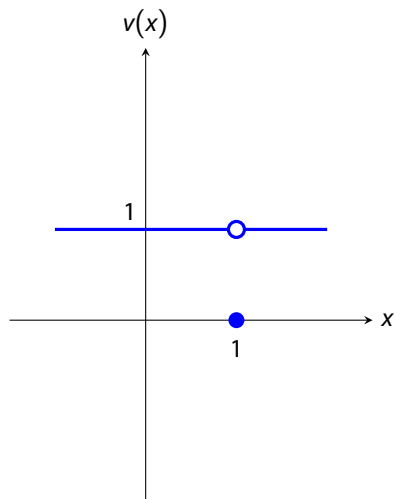
- In this example, $X = Y = \mathbb{R}$.
- Let $f(x, y) = 1 - (x - y)^2$ (a continuous function).
- Let $\Gamma(x) = [x - 1, x + 1]$, for all $x \neq 1$, and $\Gamma(1) = \{0\}$. Here Γ is lhc everywhere, but it is not uhc at $x = 1$.
- The maximum value function is $v(x) = 1$ for all $x \neq 1$, $v(1) = 0$.
- The failure of Γ to be uhc at 1 is due to an **implosion** of the correspondence at this point, with the result that less feasible points are available, so the maximum cannot be reached: there is a **downward** jump, so the value function is lsc, but not usc at $x = 1$.
- Had f been lsc, then we would still get the same result: v may have downward, but no upward, jumps.

LSC of the Value Function: Graphical Representation

Feasibility correspondence



Maximum value function

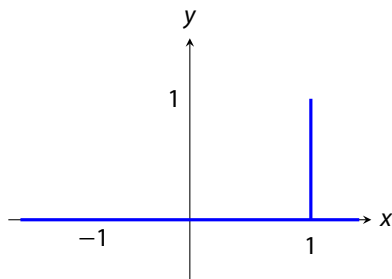


Intuition of the upper semicontinuity of the maximum values

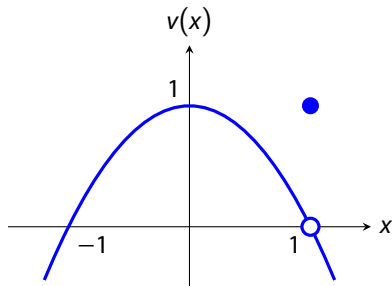
- In this example, $X = Y = \mathbb{R}$.
- Let $f(x, y) = 1 - (x - y)^2$ (a continuous function).
- Let $\Gamma(x) = \{0\}$, for all $x \neq 1$, and $\Gamma(1) = [0, 1]$. Here Γ is uhc everywhere, but it is not lhc at $x = 1$.
- The maximum value function is $v(x) = 1 - x^2$ for all $x \neq 1$, $v(1) = 1$.
- The failure of Γ to be lhc at 1 is due to an **explosion** of the correspondence at this point, with the result that more feasible points are available, so the maximum can be better approached: there is an **upward** jump, so the value function is usc, but not lsc at $x = 1$.
- When Γ is both uhc and lhc, then we have neither explosions nor implosions, so we get continuity of the value function. Note that this happens in both examples at all points $x \neq 1$.

USC of the Value Function: Graphical Representation

Feasibility correspondence



Maximum value function



Noncontinuity of policy selections

- Let $X = [0, \infty)$, $Y = \mathbb{R}$, $f(y) = y^3 - y^2$, and $\Gamma(x) = [-x, x]$.
- The assumptions of the Theorem of the Maximum are satisfied.
- The value function is:

$$v(x) = \begin{cases} 0, & \text{if } x \leq 1; \\ x^3 - x^2, & \text{if } x \geq 1. \end{cases}$$

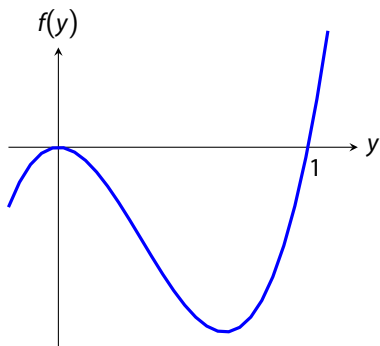
- The policy correspondence is:

$$G(x) = \begin{cases} \{0\}, & \text{if } x < 1; \\ \{0, 1\}, & \text{if } x = 1; \\ \{x\}, & \text{if } x > 1. \end{cases}$$

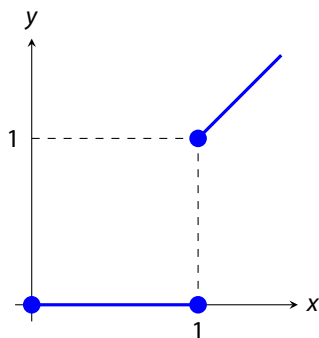
- The policy correspondence is uhc, but **it has no continuous selection**. The jump in the policy correspondence is caused by the nonconcavity of the objective function.

Noncontinuity of policy selections: Graphical representation

Objective function



Policy correspondence



Minima of usc and lsc functions

The parallels for minimization problems of the results we have seen are easy to derive from them, taking into account that

$$\sup \{f(x, y) : y \in \Gamma(x)\} = -\inf \{-f(x, y) : y \in \Gamma(x)\},$$

and **duality**.

Lemma (Semicontinuity of the value function)

Let $f : X \times Y \rightarrow \mathbb{R}$ be a real-valued function, and let $\Gamma : X \rightrightarrows Y$ be a correspondence (with nonempty values). Define

$$v(x) = \inf \{f(x, y) : y \in \Gamma(x)\}.$$

- 1 If f is lsc and Γ is compact-valued and uhc, then v is lsc and real-valued.
- 2 If f is usc and Γ is lhc, then v is usc and takes values in $[-\infty, +\infty)$.

Minima of continuous functions

Theorem (The Theorem of the Minimum)

Assume:

- 1 $f : X \times Y \rightarrow \mathbb{R}$ is a continuous function.
- 2 $\Gamma : X \rightrightarrows Y$ is a continuous correspondence with (nonempty and) compact values.

Define

$$v(x) = \min \{f(x, y) : y \in \Gamma(x)\}.$$

Define also

$$G(x) = \arg \min \{f(x, y) : y \in \Gamma(x)\} = \{y \in \Gamma(x) : f(x, y) = v(x)\}.$$

Then $v : X \rightarrow \mathbb{R}$ is a continuous function and $G : X \rightrightarrows Y$ is an uhc correspondence with (nonempty and) compact values.